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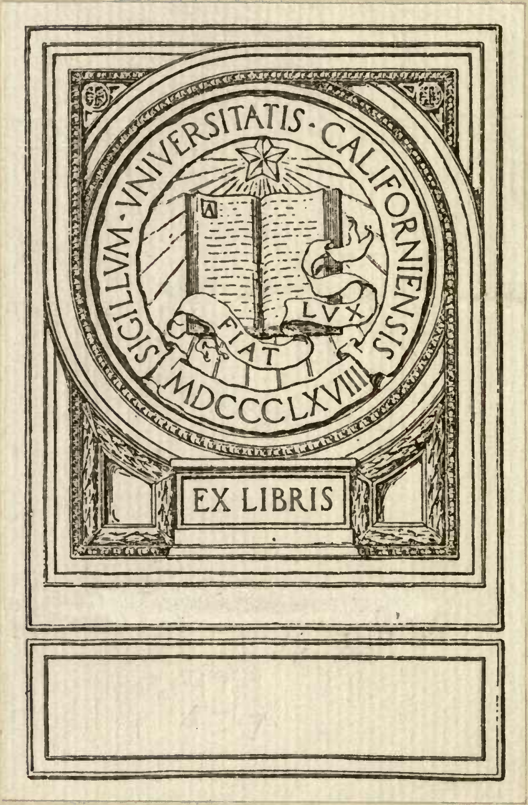
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UNIV. OF

MECHANICS

A SCHOOL COURSE

BY

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ASSISTANT MASTER AT ETON COLLEGE

FIFTH IMPRESSION

LONDON
EDWARD ARNOLD
41 & 43 MADDOX STREET, BOND STREET, W.

laboratory work is outlined in the Appendix. This course is also adapted for younger students who are beginning the subject in the laboratory. These may find it necessary to begin with Statics; in which case Chapters VI–IX should follow immediately after Chapter I.

Chapter III is occupied chiefly with Galileo's work. It may be omitted on a first reading; but apart from the historical interest, Galileo's methods were so simple and accurate, his mathematics so sound and his arguments so lucid, that they ought to be more familiar. Experiments which resulted in the discovery of the laws of motion ought to be worth repeating.

The difficulty experienced by some boys in acquiring geometrical notions of such things as angles, areas, volumes, has disappeared since these have been made subjects for measurement. In the same way it will be found that velocities, accelerations, moments, work, and momentum can be made clear to the student if he has to measure them. For those possessing a mathematical imagination, very little of this measuring may be necessary; but for others the work may take a considerable time. The writer's thanks are due to Mr. W. C. Fletcher, of the Education Department, for his kind permission to introduce his apparatus for measuring accelerations. This is an adaptation of the tuning-fork and smoked-glass method; but it is very easily adjusted, and its working is visible at a distance. It can be made to give accurate results. However, its chief value lies in its making of an acceleration something that can be seen and measured. Squared paper methods are employed freely; and no apology is offered for the tentative introduction of the Differential Calculus. The notions of variable velocity and acceleration led to the invention of fluxions; and it seems a waste of energy to separate the acquirement of these notions from the beginnings of the Calculus. Boys will always find difficulties in stating their units, and the graphical work shows up these difficulties. It is impossible to shirk them, and space, velocity, and acceleration graphs must be plotted until they are overcome.

An experimental verification of Newton's Second Law has been given. To justify this rash act it has been necessary to state the law in Newton's own words, 'Change of motion *is proportional to* the impressed force,' and the force has been measured in gravitational units. The dyne, and, if need be, the poundal, can then be defined, and recognized as not differing in kind from other forces of a more familiar type.

The measurement of work forms an important part of the book. Work is made the basis for the treatment of machines. The experiments become much more of a reality when this course is followed. The efficiency of a machine is the important feature, and the difference between theoretical and practical 'mechanical advantage' should be emphasized rather than slurred over. It should not be disconcerting to find that a system of demonstration pulleys in equilibrium remains so when P is doubled.

The influence of Professor Perry's work in general, and of his *Applied Mechanics* in particular, is gratefully acknowledged.

The author takes this opportunity of thanking Mr. C. E. Ashford for much valuable criticism and help. His thanks are also due to his pupils, Messrs. E. R. Campbell, H. G. Farmer, and R. C. Bourne, who have worked through many of the exercises.

Many of the examples are taken from recent Army and Navy papers by kind permission of the Controller of H. M.

Stationery Office. Some others are taken from London Matriculation papers.

The five-figure logarithm tables are extracted from Mr. J. B. Dale's useful book; but the natural trigonometrical ratios are given to three places only. This degree of accuracy is sufficient in most of the practical work.

The author will be grateful for any corrections.

W. D. E.

ETON, *March*, 1905.

NOTE TO FIFTH IMPRESSION

A SECOND Appendix has been added, containing a selection of questions in Practical Mechanics, taken, by permission, from recent Army Examination papers.

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MECHANICS

CHAPTER I

ELEMENTARY NOTIONS OF FORCE

§ **1. Muscular Sense.** Lift a heavy weight by the handle, or by a loop of thick string tied on to it. You will be conscious of a **pressure** of the string on your fingers, and of a muscular strain, a **tension** in the muscles of your arm. The sense of touch and what is called the *muscular sense* will give you some rough estimate of the weight you are lifting; and with constant practice you might be able to judge weights fairly accurately. But, as in the sensation of heat and cold, the senses are not very trustworthy guides, and the muscles are affected by fatigue and conditions of health and training. To measure with, we want something that does not change its condition from day to day. A steel spring is useful in this way. The temperature may affect it slightly, and even metals are liable to what Lord Kelvin has called 'elastic fatigue': but a good spring-balance or dynamometer gives a fairly accurate record.

§ **2. To check the accuracy of a Spring-balance.** Take a spring-balance and a set of weights corresponding to the graduations of the balance. Weigh a scale-pan and hang it from the hook of the balance. Note the reading of the balance. Then put weights into the pan, increasing them gradually, and note the reading of the balance in each case. Then take off the weights gradually, noting the readings. Enter your results thus :—

Weight suspended.	Balance Readings.	
	(i) Loading.	(ii) Unloading.

It is advisable that the student should make a spiral spring for himself and test whether its extension is proportional to the stretching force. Steel wires are best wound on a lathe; but a thin steel wire can be wound fairly well on a straight piece of glass tubing. It should then have one end held in a clamp, and the other attached to a light scale-pan, the weight of which must be found.

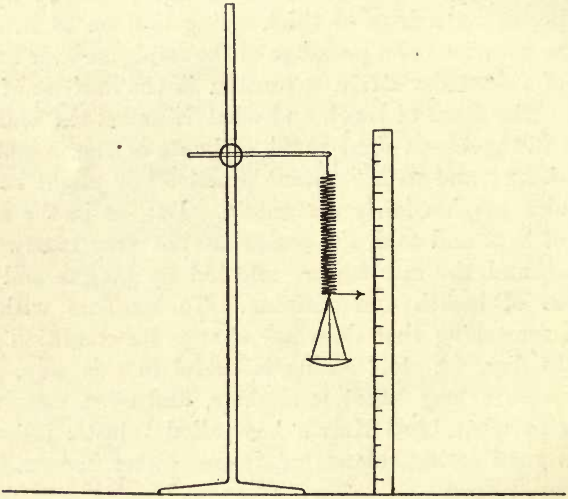

FIG. 1.

Fig. 1 shows the arrangement. A metre scale can be fixed vertically by the side of the spring to measure the extension. Compare the readings, loading and unloading, and tabulate them as before. Plot a curve with extensions as ordinates, and loads as abscissæ. *Do you find any limit to the elasticity of the spring?* In other words, is there any load from which the spring does not recover?

Make another spring by coiling a close spiral of copper wire, and test its limit of elasticity, beginning with very small weights. No. 17 wire is suitable for this experiment. Plot a curve as before.

§ **3. Spring-balances in series and in parallel.** Take two spring-balances and hook one on to the other (tandem fashion), and hang a weight from the lower of the two. Are the readings of the two balances the same? Can you account for any difference? The same experiment may be repeated with three or more in 'tandem,' or 'series' as it would be called in speaking of electric cells. Now try the same weight with two balances in parallel. Fig. 2 shows a simple way of arranging this. Are the readings in this case the same as before?

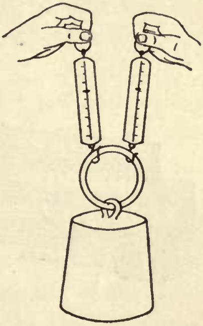

Fig. 2.

§ **4. Tension and Pressure.** The spring of the balance is extended by a *tension* or stretching force. There are also spring-balances which register compressing force or *pressure*. One of these should be tested in a similar way (see Appendix). Having made himself familiar with the measurement of tensions and pressures, the student should endeavour to measure attractions and repulsions. The attraction of a horseshoe-magnet on its keeper, when the latter is in contact with it, may be found by hanging weights to the keeper until it just comes off. Magnetic repulsion of two like poles can be measured with very fair accuracy by Hibbert's magnetic balance (see Appendix), in which the repulsion of one magnetic pole on another like pole is balanced by a sliding weight (Fig. 3).

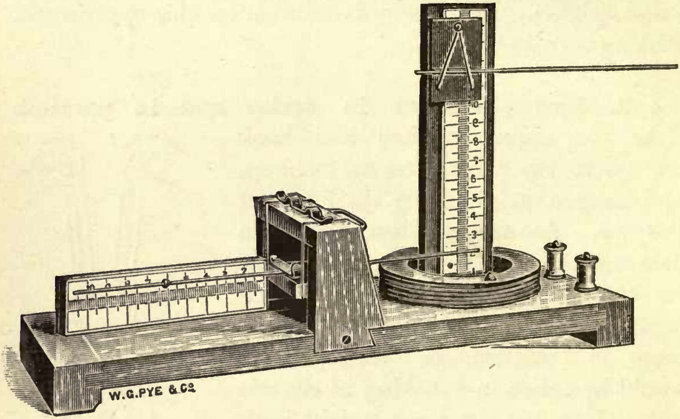

Fig. 3.

Attractions and repulsions are of the nature of tensions and pressures in an invisible medium. All bodies are pulled towards the earth, drawn as it were by invisible cords, of which we know no more than we do of the attractions and repulsions of magnets. The same attractive force keeps together the sun and the planets and their moons. 'Action at a distance,' without any intervening medium, was regarded by Newton as inconceivable, and we may not be far wrong in thinking that every force is either a push or a pull, a pressure or a tension, or both.

§ **5. The Earth's Attraction.** When we lift a heavy weight from the ground, it is difficult to judge whether our muscles are more tightly strained during the act of lifting than when we are simply preventing the weight from falling. The spring-balance will here be useful. Hook the balance on to the weight, and lift the weight by the balance, noting at the same time whether the balance reading is more during the act of lifting than when the weight is simply hanging from the hook. Does it make any difference whether the weight is lifted quickly or slowly? Again see whether the reading of the balance alters when the weight is lowered: also whether the rate of lowering makes any difference.

When the weight is hanging from the spring-balance, it is acted on by two equal and opposite forces, the tension of the spring pulling it up, and the attraction of the earth, which we call the **force of gravity**, pulling it down. If the tension is increased, the weight begins to move upwards, and if it begins to move downwards, the tension is diminished. When you stand on a weighing machine, e. g. the automatic penny-in-the-slot machine, your true weight will not be recorded unless you stand still. If you give a jump, you send yourself upwards with a force greater than your own weight; and as you obtain this force from the pressure of the soles of your feet on the footplate of the machine, the pointer is driven further round. Then, as you reach the top of your spring, the pointer swings back to less than the true reading.

A similar effect may be noticed in a passenger lift. If a weight is hung from a spring-balance in a lift, or if a person stand on a spring weighing-machine carried by the lift, it may be noticed that the reading of the spring is increased while the lift is starting upwards, and diminished while the lift is starting downwards; also that, when the lift has reached its maximum speed and is moving steadily, the reading of the spring returns to the normal, but increases again as the lift slows up on the downward journey, and diminishes as it begins to come to rest on the upward journey.

It is not always possible for the student to make experimental measurements of this kind; but he should remember to study the sensations of the soles of his feet the next time he stands in a lift, even if it does not carry an automatic weighing-machine.

§ **6. Weight.** The **weight** of a body is the force with which the earth attracts it, and may be measured by the pull of a spring-balance when the body is at rest or moving with uniform speed.

Is this force different at different places? If you made accurate measurements of the earth's magnetic force at different points in the room in which you work, you might very likely obtain results differing by as much as five or ten per cent., more particularly if there happened to be much iron in the room. But if you hang a 'weight' from a delicate spring-balance, you will not be able to notice any difference in the reading at different places, even if you go up to the top of the house. It does not follow that the weight is the same everywhere. It is not. Suppose you had a spring-balance sensitive enough to show single grammes up to 1 kilogramme (1,000 grammes). Imagine a kilogramme 'weight' to be hung from this balance, the reading of the balance being of course 1,000. Now suppose yourself, balance and all, to be removed and established at a height of two miles above sea-level. The pointer should then show 999, for the force of the earth's attraction on the kilogramme would be less by 1 in 1,000.

The experiment, though not impossible, is beyond the reach of most of us; and while we are exercising our imagination, we may as well let it carry us to a height of 4,000 miles, so that our distance from the centre of the earth is doubled. (Earth's radius about 4,000 miles.) Our spring balance would then show 250 only, and the force of gravity would be one quarter of its value on the earth's surface. A man who could hold 56 lb. at arm's length would with the same muscular force be able to hold 2 cwt., if he were placed at a distance of 8,000 miles from the earth's centre.

Even on the surface of the earth itself the force of gravity varies. For instance, a spring-balance recording 1,000 grammes in our latitudes would record at the equator about 998 grammes, with the same kilogramme weight attached. This discrepancy is due in part to the shape of the earth, which is not perfectly spherical.

It follows that a spring-balance manufactured for use in weighing out quantities of butter or other material in London will not give quite the same quantities when used in other latitudes. As a matter of fact, spring-balances are not used when great accuracy is required, and an accuracy of one per cent. is as much as can be expected from them. Therefore a difference in weight of one or two in a thousand would not be apparent. It is quite a different thing with ordinary balances which are capable of much greater accuracy than one in a thousand. Will a good chemical balance manufactured in Holland be of any use in Edinburgh or Madras? Yes, for the weight in one scale-pan is compared with a standard 'weight' in the other scale-pan, and, if one has diminished, the other will have diminished equally. Thus a quantity of sugar which counterpoises a standard pound or gramme 'weight' in London will also counterpoise the same 'weight' at any other place on or off the earth's surface. The accuracy depends on the construction of the balance, and requires that the knife edges which support the two scale-pans should be at equal distances from the knife edge on which the balance turns.

Notice that the word *weight* is used sometimes to mean the lump of metal which we use in weighing, and sometimes the force with which the earth attracts that lump. There is usually no doubt as to which is meant; but it may be useful to agree that 'weight' in inverted commas signifies the lump of metal. That lump is the same lump wherever it may find itself; but its weight alters with its position. So we have this fact, that the same 'weight' may have different weights in different places. The weight alters with the position; but there is one quality intimately connected with the weight, but remaining the same, like the colour and the shape. We call it the **mass**.

§ **7. Mass.** It is not very easy to form a clear notion of a 'mass' apart from its weight. We cannot get away from the earth's attraction, and we almost always measure a mass by weighing it; for the weight is proportional to the mass: e.g. 2 lb. of sugar contains twice as much sweetening matter as 1 lb. of sugar and requires twice as much force to lift it. Is mass, then, the same as quantity of matter? We might say this if we knew matter to be all of one kind. One pound of sugar contains the same quantity of sugar as another pound of sugar; but in our present state of knowledge we can hardly say for certain that 1 lb. of sugar contains the same quantity of matter as 1 lb. of lead. At any rate, it has the same mass (and the same weight, if sugar and lead are both at the same place).

Is there any other way of measuring a mass besides the method of weighing?

There is no method so convenient or so accurate, but there are various ways depending on the action of force on mass which it is important to consider.

§ 8. **How can we examine a Mass without the interference of its weight?** The weight pulls it towards the centre of the earth, that is, vertically downwards. If the mass rests on rollers or wheels or on a smooth surface, its weight will pull it down a slope. But if the surface be perfectly horizontal, there is no reason why it should slip one way more than another. A mass resting on a smooth horizontal surface, or on smoothly running wheels on a horizontal track, is not prevented from moving by its weight.

Horses are frequently employed at railway stations for shifting trucks on level parts of the line. The horse pulls hard for a few seconds and the truck gradually gets up speed. Supposing that the horse pulls always with its full strength, we might compare the times of getting up speed when the loads are altered. The horse is then detached and the truck goes on by itself. How is it stopped, or will it stop of itself?

Notions of mass can be acquired by pushing people of different sizes in a swing. Their weight always tends to pull them back to the position in which the ropes of the swing are vertical; but if the ropes are long, we can compare the masses of two people by comparing the distances to which equal pushes will send them.

A person weighing 10 st. is sitting in a swing. A slight push sends him 6 inches forwards. How far would the same push send a person weighing 12 st.? This is a question which we might answer correctly by the light of nature, but there would be some difficulty in performing the experiment satisfactorily; for the pushes must not only be of equal strength, but must last for equal times.

The apparatus of Fig. 27 can be easily adapted for comparing roughly the effects of different forces on a mass. *W* is a roller resting on a surface which can be inclined at any angle, but which is horizontal in this case. The string passing over the pulley is attached to a very light scale-pan. The scale-pan must be very light indeed if its weight is not enough to overcome the friction of the axle, and start *W* rolling. Find out what weight is just not sufficient to do this, then add a small weight to the scale-pan and observe what happens. Does the speed of *W* increase all the time that it is moving? What is the effect of putting a larger weight into the scale-pan? Time *W* from end to end with different excess weights in the scale-pan (§§ 41, 44).

§ **9.** The following questions you may be able to answer from your own experience :—

(i) You are standing up in a railway carriage facing the engine, and the train (*a*) starts suddenly, (*b*) stops suddenly. What happens to you in each case?

(ii) You are standing up in a railway carriage and the train runs round a curve. You are impelled towards one window. Is this window on the outside or the inside of the curve?

(iii) A cart is driven rapidly round a corner, and is so near upsetting that it goes round on one wheel. Is it the outside or the inside wheel ?

(iv) What precautions do you take when getting out of an omnibus or a train that is in motion, and what would happen if you did not take those precautions ?

(v) You are sitting in the back seat of a dogcart with your back to the horse, which is trotting, and you take the opportunity of fastening your bootlace. The horse hears the train and suddenly starts galloping. You are landed in the road and your head is contused. Which part of your head probably suffers ?

(vi) What have you to do to send yourself along and to stop yourself when skating ? when running or walking ? What conditions of the surface of the ground and of the soles of your shoes are most favourable ?

(vii) The following records are taken from *Whitaker's Almanack* :—

100 yards.	Running	9⅗ sec.	Skating	9⅘ sec.
200 yards.	,,	19½ sec.	,,	17⅖ sec.
½-mile	,,	1 min. 53⅖ sec.	,,	1 min. 20⅖ sec.
1 mile.	,,	4 min. 12¾ sec.	,,	2 min. 41⅕ sec.
10 miles	,,	51 min. 5⅕ sec.	,,	31 min. 11⅕ sec.

What explanation can you give of the differences between the two sets of figures ?

(viii) Why is there a tendency of the wheels of a bicycle or motor car to skid in going round a corner ? How is the same tendency counteracted in railways? on bicycle-tracks?

(ix) Can you give any reasons for the pull of the string on your hand as you whirl it round with a stone at the end of it ? Does this pull increase with the speed ? If the stone is being whirled in a vertical circle, is there any difference in the pull of the string at different points of the circle ? When is it greatest ? When least ? The performance of ' Looping the Loop ' is similar to this. What supplies the place of the string in this, and why is great speed necessary? (Remember that the words ‘centrifugal force’ explain nothing.)

(x) Describe the process of ‘throwing the hammer.’ At what point of the circle should one let go the handle so as to get the best result?

§ **10. Inertia.** In the light of the foregoing experiments and experience, you are asked to consider the following statement:—

No body (and by ‘body’ we mean any material object alive or dead, e. g. a stone, a bird, a balloon, a fish, a ship, a man, a motor car)—no body can start moving, stop, or change its course without the help of some force outside itself.

This statement is taken from Newton's First Law of Motion, and it gives the meaning of the word **inertia.**

Inertia is the inability of matter to start, stop, or turn of itself. All matter possesses this quality of inertia, and if we agree to measure inertia in quantity by the force required to produce a given change of motion, we may define the mass of a body as the **quantity of inertia** which it possesses.

§ **11. Newton's First Law.** Every body continues in the same state of rest or of uniform motion in a straight line, except in so far as it may be compelled to change that state by force from without.

CHAPTER II

VELOCITY AND ACCELERATION

§ **12. Speed.** To measure speed we must employ a unit of space and a unit of time; e.g. the speed of a railway train may be 60 miles an hour. This is the same as a mile a minute, or 1760 × 3 feet in 60 seconds, or 88 feet in one second. In all of these expressions the two units, of space and time, must be stated. The speed is **uniform speed** if equal distances are travelled in equal times. A train may go 60 miles in one hour, but unless it goes one mile in each minute, and 88 feet in each second, the speed is not uniform.

(i) Express a speed of 50 miles an hour in metres per second. (1 metre = 39·37 inches.)

(ii) A man runs 100 yards in $10\frac{2}{5}$ seconds. If he kept up this speed, how long would he take to run a quarter-mile, and a mile?

(iii) Express 10 yards a second in miles per hour, and in metres per minute.

§ **13. Displacements and velocities.** If a man walks 3 miles E., and then 4 miles N., he has walked altogether 7 miles; but he is only 5 miles from his starting-point, and his bearing is about 53° North of East from his starting-point. His **displacement** might be expressed thus—5 miles E. 53° N. Prove this by drawing the figure to scale, and observe that to know the displacement we must know *direction* as well as *distance.*

(i) Express as above a displacement of 2 miles E., 2 miles N.

(ii) A man walks 5 miles N., then 7 miles NE. Another man starting from the same point walks 7 miles NE., then 5 miles N. Draw these walks to scale, and find the displacement.

It will be seen from these examples that we can represent two displacements by two straight lines, and that the resultant displacement is another straight line forming the third side of the triangle if we draw the displacements successively, or the diagonal of the parallelogram, if we draw them both from the starting-point.

Velocity is rate of change of displacement; and to know the velocity of a moving point we must know the *direction* in which it is moving, as well as the *speed.* Thus a train running round a curve may be moving with uniform *speed*; but the velocity is changing because the direction is changing. A body is moving with uniform velocity when it moves continually in the same straight line and describes equal distances in equal times.

It is plain that if a point has a velocity v, reckoned in foot-second units, then in each second the displacement is v feet, and in t seconds the displacement is vt feet. This may be expressed by the formula $s = vt$.

(iii) A train travels x miles in y minutes. What is the average speed in feet per second?

(iv) A ship is 'crossing the line' in a SW. direction at a speed of 20 knots. Draw a figure to represent the velocity, and find from your figure the rate at which the latitude is changing. (1 knot means 1 nautical mile per hour. 60 nautical miles = 1 degree of latitude.)

§ 14. **Independence of the different movements of a body.** A man is in a railway carriage at A which is going from left to right at uniform velocity. He walks across to the opposite corner of the carriage at B; but meanwhile the train has moved on, and A is at A', B at B'. So that the man finds himself at B'. If he crossed the carriage at uniform speed, he was at b when A was at a, and his actual path in space was the straight line AB'. The two movements go on independently, and the resulting movement is along the diagonal of the parallelogram, whose adjacent sides are AA', AB (Fig. 4).

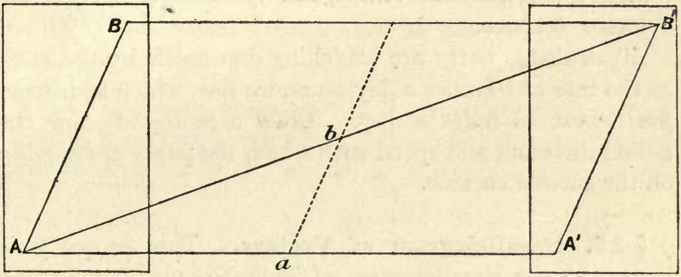

Fig. 4.

We see then from this that if a moving point have at one and the same time two different velocities which we will represent by the lines AB, AD (Fig. 5), so that one velocity would take the moving particle from A to B in one second, and the other velocity would take it from A to D in

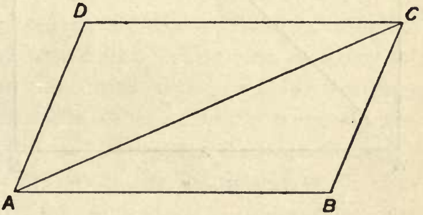

Fig. 5.

one second, then the actual velocity of the particle is represented by AC, the diagonal of the parallelogram.

(i) A man in a train going 12 miles an hour moves from one window of the compartment to the other at the rate of 5 miles an hour. Draw a figure to scale showing his actual path in space, and find from it, and by calculation, his actual velocity in space.

(ii) A sledge party are travelling due north by the stars at the rate of 10 miles a day on an ice floe, which is drifting south-west 15 miles a day. Draw a figure to show the actual direction and speed with which the party are moving on the earth's surface.

§ 15. Parallelogram of Vectors. This property is known as the Parallelogram of Velocities; and it is true not only of velocities but of all quantities which can be represented in size and direction by straight lines, such as displacements and accelerations. These quantities are known as 'vectors.' A walk of two miles to the north is a vector quantity; so is a walk of two miles to the east. The two combined are equivalent to a walk of 2·828 miles to the north-east (Fig. 6).

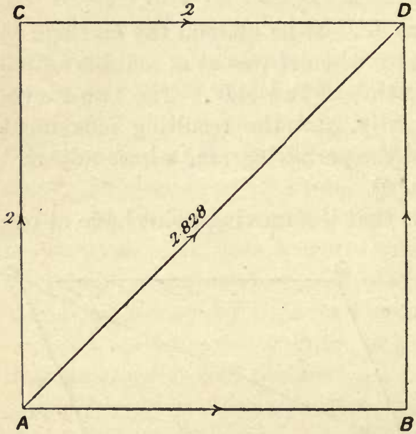

FIG. 6.

The vectors can be placed so as to form a triangle ACD, or *ABD*, or two sides of a parallelogram *ABDC*; but the result is the same in each case, viz. that the single vector *AD* is equivalent to the other two.

An experimental verification of the Parallelogram Law in the case of Forces is given later on (§ 61).

Definition. Vectors are quantities possessing direction as well as magnitude, which can be compounded by the Parallelogram Law.

§ **16. Relative velocity.** If the rain is coming down straight, a man who is standing still would hold his umbrella directly over his head; but if he is walking along in the rain, he finds he must hold it slightly in front of him to prevent the rain coming in on him. The rain appears to him to be slanting in on him: and so it is, for his walking forward has precisely the same effect as would be produced if the rain had a velocity in the opposite direction communicated to it. The velocity of the rain **relative** to the man is best found in this way. *Reduce the observer to rest* by putting on to him an equal and opposite velocity, and giving the same to the rain. Hence the relative velocity

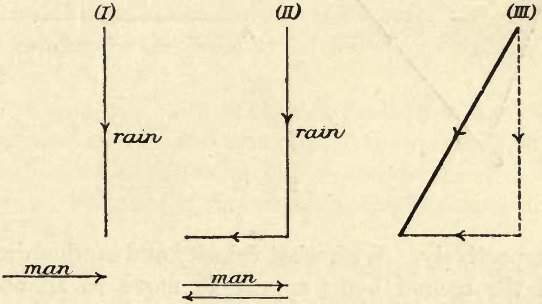

Fig. 7.

of the rain is made up as in Fig. 7 (ii), which, by the law of vectors, gives the velocity shown in Fig. 7 (iii).

The relative velocity of two moving points is found by imparting to both of them such a velocity as will reduce one of them to rest. Then the resultant velocity of the other is the relative velocity.

(i) Two bodies move with velocities represented in magnitude and direction by any two straight lines *AB* and *CD* respectively. With what velocity and in what direction would the second body appear to move to an observer stationed upon the first (Fig. 8)?

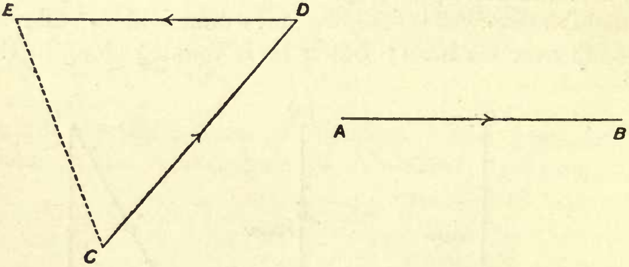

FIG. 8.

Reduce the observer on the line *AB* to rest by giving him an equal and opposite velocity. Giving the same velocity to the other body, we find that its relative motion is made up of the two vectors *CD*, *DE*, which are equivalent to *CE*.

CE therefore represents the apparent velocity in magnitude and direction.

(ii) To an observer on the deck of a steamboat travelling east with a velocity of 10 knots, the wind appears to blow from a direction 60° north of east, and with a velocity of

20 knots. Find the actual direction and velocity of the wind.

Here the apparent velocity of the wind is AB. But this is made up of two components, one of which is AC, equal and opposite to the observer's velocity (Fig. 9).

Hence the other component is CB.

CB is therefore the actual velocity of the wind, and it is obviously from due north, and at $10\sqrt{3}$ knots.

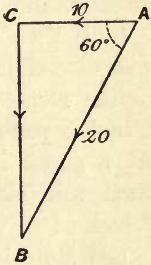

Fig. 9.

(iii) A steamer going at a speed of 8 miles per hour is steered due north in a current of 4 miles per hour flowing eastwards. Find completely the velocity of the steamer relatively to the shore.

(iv) A man in a railway train, travelling at a given rate, carries a loaded rifle, and fires it off so as to strike a telegraph post as he passes it; having given the velocity of the bullet, determine, by graphic construction, the direction in which he must aim.

(v) A vessel is steaming south-west at 5 miles an hour, and is acted on by wind and current. The wind alone would drive the vessel in a northerly direction at 4 miles an hour, and the current alone in an easterly direction at the same rate. In what direction, and with what velocity, does the vessel travel?

(vi) A train is travelling at the rate of 50 feet per second, when a rifle bullet fired at right angles to the train passes through the opposite windows of a carriage, 8 feet wide, with a velocity of 200 feet per second. Draw a picture showing the path of the bullet through the carriage.

(vii) A pistol shot is fired from a railway carriage travelling with velocity u so as to strike an object seen (at the instant of firing) in a line at right angles to the direction of motion of the carriage. Assuming that the charge of powder in the cartridge can impart a velocity v to the shot, find the direction in which the pistol must be aimed.

(viii) Two small marbles A and B are moving in clockwise direction in concentric circular grooves of 2 and 3 inches radius respectively. The velocity of A in its groove is 2 inches per second and that of B is 9 inches per second. At a given instant the marbles are 1 inch apart, what time will elapse before the distance between them is 5 inches?

(ix) Rain is falling vertically, speed 5 miles an hour, on a man walking at 4 miles an hour. At what angle to the vertical must he hold his umbrella to make the rain-drops hit the top at right angles? Give the angle to the nearest degree, determining it in any way you please.

(x) A horizontal disc rotates uniformly in clockwise direction about a vertical axis through its centre. A fly walks at a uniform rate of 1 inch in 30 seconds, along a diameter of the disc from one end to the other, and during its journey, which occupies 4 minutes, the disc rotates twice. Show, by a graphical construction not necessarily to scale, the form of the curve in which to an observer regarding it from above, the fly would appear to move.

Find the actual velocity of the fly at the instant it reaches the end of its journey.

(xi) To a passenger on a ship steaming northwards with velocity 10 miles per hour, the wind appears to blow from the north-west with velocity 14 miles per hour. Find the true direction and velocity of the wind.

(xii) One steamboat A steams north with a constant velocity of 12 miles an hour, another, B, steams east with a constant velocity of 16 miles an hour. When A is due east of *B*, it is 10 miles from it. Find the velocity of *B* relative to *A*, and the nearest distance to which the boats will approach.

§ **17. Space and Time Diagrams.** Fig. 10 gives a graphical representation of part of a railway time-table: the following extract from Bradshaw should be compared with the diagram :—

Miles			
0	London Bridge		10.56
$2\frac{3}{4}$	New Cross		11. 2
$8\frac{1}{2}$	Norwood		11.16
$10\frac{1}{4}$	Croydon (East)		11.23
$20\frac{3}{4}$	Redhill		11.41
$21\frac{1}{2}$	Earlswood		11.45
$25\frac{1}{2}$	Horley		11.53
$29\frac{1}{4}$	Three Bridges	arr. dep.	12. 1 12. 3
$33\frac{3}{4}$	Balcombe		12.12
$37\frac{3}{4}$	Hayward's Heath	arr. dep.	12.20 12.22
$49\frac{1}{4}$	Preston Park		12.40
$50\frac{1}{2}$	Brighton		12.47

Observe that the distances marked off along the horizontal axis (usually called the **abscissæ**) represent times reckoned from 10.50 a.m., each space representing ten

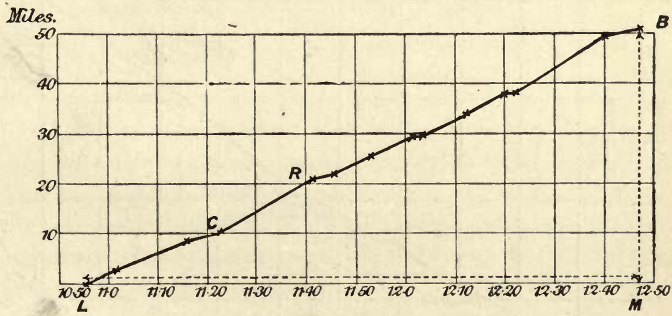

Fig. 10.

minutes. The vertical spaces represent miles, so that the vertical distances of the marked points from the horizontal line (usually called the **ordinates**) represent the distances of the train from London Bridge in miles at the particular times above which they are marked. Thus at 10.56 the train is at London Bridge and the mark is on the horizontal axis. At 12.47 the train is $50\frac{1}{2}$ miles from London Bridge. BM represents $50\frac{1}{2}$ miles and LM represents 1 hr. 51 min., the time taken in going from London Bridge to Brighton.

$$\frac{BM}{LM} = \frac{50\frac{1}{2}\text{ miles}}{111\text{ minutes}} = \frac{\cdot455\text{ mile}}{1\text{ minute}} = \frac{27\cdot3\text{ miles}}{60\text{ minutes}}.$$

Hence the average speed of the train is 27.3 miles an hour, and if the speed were uniform throughout the journey, the intermediate points would lie on the straight line joining LB. It is obvious that they do not lie on this line, and that the speed is not uniform. The student should plot this curve for himself on squared paper, taking $\frac{1}{10}$ inch for each mile and each minute.

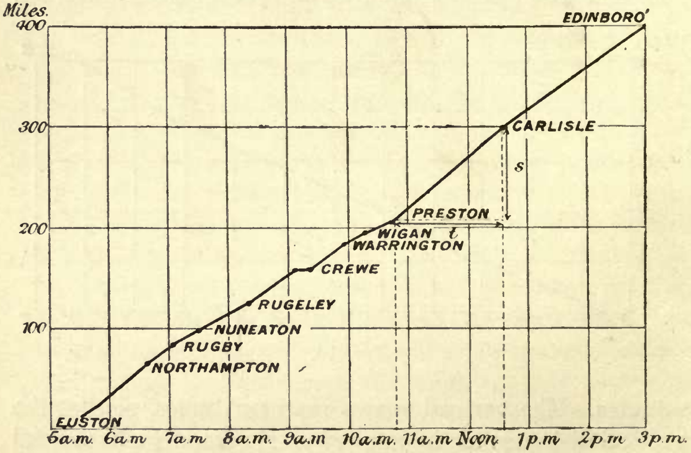

FIG. 11.

(i) Between which stations is the speed (*a*) greatest? (*b*) least? In two instances the line joining two points is horizontal. What does this mean? Does the straight line *CR* give a complete representation of the speed of the train between Croydon and Redhill? If not, what kind of line would show this more exactly?

(ii) Fig. 11 shows another diagram for a London and North-Western Railway train leaving Euston 5.15. From this diagram determine (*a*) the average speed between London and Edinburgh, (*b*) the average speed between Preston and Carlisle.

(iii) Plot on squared paper the following time-tables :—

Miles			
0	Margate Sands		8. 0
3¾	Ramsgate Town		8.12
8¼	Minster		8.20
19½	Canterbury		8.40
33½	Ashford	arr.	9. 3
		dep.	9. 5
41¾	Paddock Wood		9.37
75½	Cannon Street		10.30
75¾	Waterloo Junction		10.37
76¼	Charing Cross		10.41

Miles			
0	Euston		6.15
5½	Willesden Junction		6.28
11½	Harrow		6.39
17½	Watford		6.51
28	Berkhamsted		7.17
31¾	Tring		7.26
40¼	Leighton		7.45
46¾	Bletchley	arr.	7.55
		dep.	8. 8
82½	Rugby		8.50

§ **18.** A true space and time chart would show the gradual slowing down of a train coming into a station by the straight line of uniform velocity becoming a curve which would slope less and less until it became horizontal when the train stopped. Then on starting again the curve would gradually return to its maximum slope again. The shape of the curve near a station is indicated in Fig. 12, in which the interval of time *MM′* shows the duration of the stoppage. Now imagine that the curve in Fig. 13 represents the starting of a train. At the end of a minute the curve has become a straight line, and the velocity is 60 miles an hour

$\left(\frac{RS}{PS}=\frac{1 \text{ mile}}{1 \text{ minute}}\right)$. To begin with the velocity was zero;

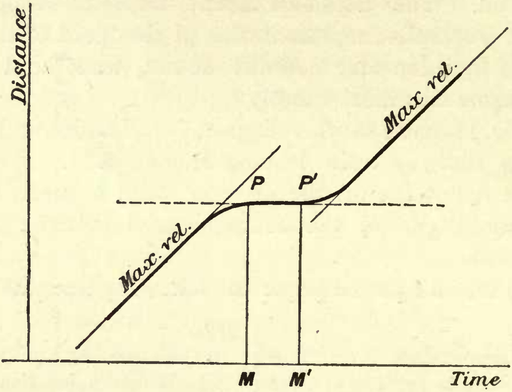

FIG. 12.

but how are we to find the velocity at the end of the first half minute? Obviously if the speed at Q had been the maximum the curve would have continued in the straight line QT', i.e. in the **direction of the tangent** to the curve at Q. Hence the velocity at Q is $\frac{QN}{NT}$, which we may call the **slope** of the curve at Q, and which is tan ϕ, where ϕ is the angle which the direction of the curve at Q makes with the axis of x.

The tangent line to a curve at a point P is obtained by taking another point P' on the curve (Fig. 14) and joining to P, and then imagining the final position of this line when P' moves up to coincide with P. (See any modern textbook of geometry.)

The direction of the curve at any point is the direction of the tangent line at that point. We see then that the velocity at any point in our space and time diagram may be obtained by drawing the tangent line at that point and finding its slope, in other words, the tangent of the angle that it makes with the horizontal line [1].

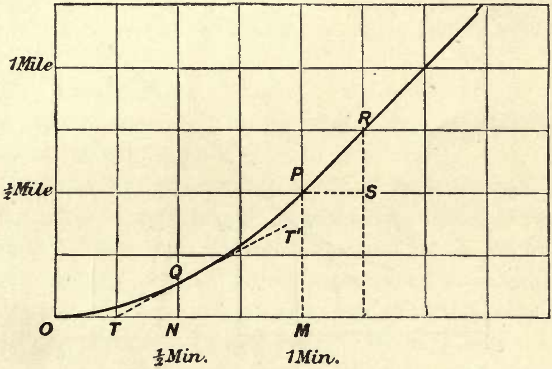

FIG. 13.

(i) What is the velocity of the train at the end of the first half-minute (Fig. 13)?

(ii) A train moves through 100 yards in $4\frac{3}{5}$ seconds, a quarter of a mile in 20 seconds, and half a mile in 39 sec. Plot a space and time graph. How can you tell from this whether the velocity is uniform or not?

(iii) A stone falls through 1 foot in $\frac{1}{4}$ second, 4 feet in $\frac{1}{2}$ second, 9 feet in $\frac{3}{4}$ second, and 16 feet in 1 second. Plot a graph and obtain from it as nearly as possible the velocity at the end of each quarter-second.

---

[1] Note that it is the ratio $\frac{\text{perpendicular}}{\text{base}}$ which is to be measured for this line, and these must be measured in whatever units have been employed. For instance, it may be found convenient to take 10 spaces as horizontal unit, and 1 space as vertical unit, and this must, of course, be taken into consideration. This point will be made clearer by a careful examination of Fig. 15.

(iv) Plot the curve $s=80t-16t^2$, and show from your graph that the velocity at the end of $2\frac{1}{2}$ seconds (i.e. when $t=2\frac{1}{2}$) is zero.

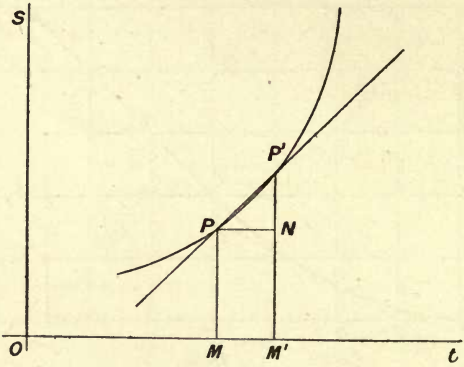

FIG. 14.

[This may be shown in another way. Take a point P on the space and time diagram (Fig. 14); let the time represented by OM be called t, and the space represented by PM be called s. Then if P' be another point near P, the small increment of time MM' may be called δt, and the corresponding increment of space $P'N$ may be called δs. The slope of the line PP' is obviously

$$\frac{P'N}{PN}=\frac{\delta s}{\delta t}.$$

When P' coincides with P, the line PP' becomes the tangent line, and the limiting value of the ratio $\frac{\delta s}{\delta t}$ gives the actual velocity at the point P. It is then written $\frac{ds}{dt}$. Let $(s+\delta s,\ t+\delta t)$ be the space and time corresponding to a point on the graph near the point (s, t). See Fig. 14.

Then

$$\begin{aligned} s &= 80t-16t^2 \\ s+\delta s &= 80(t+\delta t)-16(t+\delta t)^2 \\ &= 80t+80\delta t-16t^2-32t\delta t-16(\delta t)^2. \end{aligned}$$

By subtraction we get

$$\delta s = 80\delta t-32t\delta t-16(\delta t)^2$$

$$\frac{\delta s}{\delta t} = 80-32t-16\delta t.$$

Now when we make the two points P and P' (Fig. 14) coincide, δt becomes zero, and $\frac{\delta s}{\delta t}$ becomes $\frac{ds}{dt}$, the velocity at P.

$$\therefore \frac{ds}{dt} = 80-32t.$$

Find from this formula, and from the graph, the velocity at the end of 0, 1, 2, $2\frac{1}{2}$, 3, 4, 5 seconds.]

§ 19. Acceleration.

The acceleration of a moving point is the rate of change of its velocity.

It is found by experiment that, when a stone falls, its velocity after a quarter of a second is about 8 feet per second. At the end of half a second, it is 16 feet per second. At the end of three-quarters of a second it is 24 feet per second, and at the end of one second it is 32 feet per second. Plot these values in a velocity-time graph, and the graph will be found to be a straight line. In each quarter-second the velocity has increased by 8 (ft. per sec.). And in one second the increase of velocity is 32 (ft. per sec.). The measure of the acceleration in foot-second units is 32.

(i) With the same acceleration, what is the velocity after $1\frac{1}{4}$ seconds? after $3\frac{1}{2}$ seconds?

If the acceleration is a, then in t seconds the velocity will have increased by an amount at.

Suppose that the velocity is u to start with, then if v is the velocity at the end of t seconds $v = u+at$.

Space described with uniform acceleration.

Divide the time t into n equal intervals, each interval being τ, so that $t = n\tau$.

Suppose that during each interval, the velocity remains equal to its value at the beginning of that interval. The velocities at the beginnings of the intervals are

$$u,\ u+a\tau,\ u+2a\tau,\ \&c.$$

and the space described is

$$u\tau+\tau(u+a\tau)+\tau(u+2a\tau)+\dots+\tau(u+\overline{n-1}\,.\,a\tau)$$
$$=nu\tau+a\tau^2(1+2+3+\dots+(n-1))$$
$$=ut+a\tau^2\frac{n(n-1)}{2}=ut+\tfrac{1}{2}at^2\left(1-\frac{1}{n}\right).$$

Now if during each interval the velocity remained equal to its value at the end of that interval the space described would be

$$\tau(u+a\tau)+\tau(u+2a\tau)+\dots+\tau(u+na\tau)$$
$$=ut+\tfrac{1}{2}at^2\left(1+\frac{1}{n}\right).$$

The true space described is intermediate between these two values, and if we make n very large, so that each interval τ is very small, the two values approach one another, and finally when n is infinite, $\frac{1}{n}=0$, and each value becomes $ut+\frac{1}{2}at^2$.

The formula $s=ut+\frac{1}{2}at^2$ must be remembered; and Figs. 21 and 23 should be examined in connexion with the above proof.

(ii) From the formulæ $v=u+at$

$$s=ut+\tfrac{1}{2}at^2$$

eliminate t and obtain the formula

$$v^2=u^2+2as.$$

(iii) A body starting from rest and moving with uniform acceleration describes 100 feet in 5 seconds. Find the acceleration, the velocity at the beginning and the end of the fifth second, and the space described in the fifth second.

(iv) A train starting from rest gets up a velocity of 60 miles an hour in two minutes. What is the average acceleration (*a*) in mile-minute units, (*b*) in foot-second units?

When the velocity is decreasing, the acceleration is negative. Thus taking the acceleration of a falling body as 32 in foot-second units, we find that a stone thrown up straight with a velocity of 96 feet per second will have lost all that velocity after 3 seconds. The foregoing equations might, in the case of bodies thrown up straight with an initial velocity u, be written

$$v = u - gt$$
$$s = ut - \tfrac{1}{2} gt^2,$$

a being replaced by $-g$, the retardation due to gravity.

(v) Find the greatest height attained by a stone thrown straight up with a velocity of 112 feet per second. (The velocity at the highest point is momentarily zero. Assume $g = 32$.)

(vi) Find the time taken by the stone on the return journey, and its velocity when it reaches the ground again.

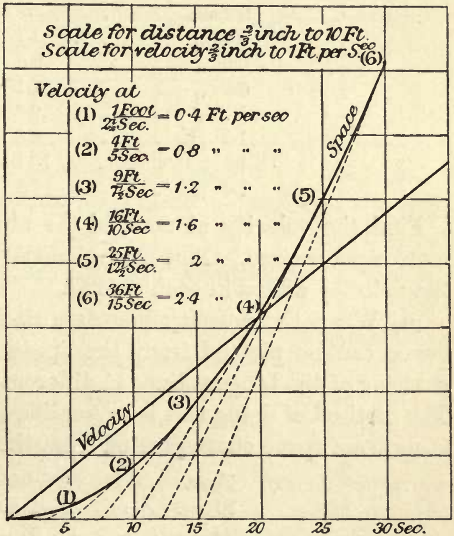

FIG. 15.

§ 20. Fig. 15 shows a space-time graph from which has been plotted a velocity-time graph. This has been done by finding the slope of the curve at various points, and plotting the values at intervals corresponding to those of the first graph. It will be noticed that the slope of the second graph is uniform, and the speed increases uniformly. In other words the acceleration is constant, and equal to $\frac{\cdot 4 \text{ ft. per sec.}}{5 \text{ sec.}} = \cdot 08$ ft. per sec. per sec. This means that in every second the speed increases by an amount equal to ·08 ft. per second (or 4·8 feet per minute). This is called an acceleration of ·08 (ft.-sec. units).

(i) Plot a space-time curve from the following data :—

Time.	*Distance.*
0·2 sec.	·5 cm.
0·4 „	1·1 „
0·6 „	2·3 „
0·8 „	4·1 „
1· „	6·5 „
1·2 „	9·5 „
1·4 „	13·1 „
1·6 „	17·3 „

Find the velocity at each of the above points in centimetres per second. Plot the velocity-time graph, and find the velocity added in each second.

(ii) When the velocity increases uniformly, the velocity curve can be plotted from the space curve without the drawing of the tangent lines at different points of the curve. The method of doing this is exemplified in Fig. 16, in which is shown a space curve plotted from the following data :—

Time.	*Distance.*
10 sec.	2 feet.
20 „	8 „
30 „	18 „
40 „	32 „

Now since 2 feet are travelled over in the first 10 seconds the average velocity during those 10 seconds is 0·2 foot per second, and if the increase of velocity is uniform, this will be the actual velocity at the end of 5 seconds. Hence, taking 1 inch to represent 1 foot per second, we erect at 5 seconds an ordinate to represent 0·2 foot per second. The average velocity during the second 10 seconds is 6 feet per 10 seconds, i.e. 0·6 foot per second. Hence an ordinate is

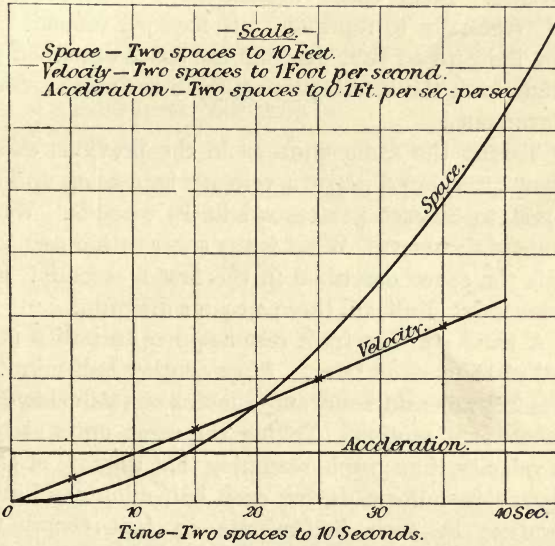

FIG. 16.

erected of 0·6 inch at the 15 seconds mark; and so on. The points so obtained lie on a straight line, showing that the velocity does increase uniformly. If it did not this method would be only approximate.

Apply this method to the space curve obtained in the previous exercise, and compare the result with the velocity curve obtained in the other way.

The acceleration curve is obtained from the velocity curve in a similar way, and as in this case the acceleration is constant, the curve is a straight line parallel to the time-axis.

(iii) Taking times as abscissæ, and velocities as ordinates, represent on squared paper a uniform velocity of 10 feet per second. Each space horizontally to represent one second, vertically to represent one foot per second. How can you indicate on your diagram the space described (*a*) in 5 seconds from the starting-point, (*b*) in the ninth second of the movement?

(iv) Taking the same units as in the previous exercise, represent on squared paper a velocity increasing uniformly from rest to 20 feet per second in 10 seconds. What is the slope of the curve? What is the acceleration (see p. 30)? What is the space described in the first 5 seconds? in the last 5 seconds? Indicate these on your diagram.

(v) A train starting from rest acquires in half a minute a speed of 15 miles an hour. After another half-minute the speed is $22\frac{1}{2}$ miles an hour, and remains at this value during the subsequent motion. Taking the same units as before plot a velocity-time graph, assuming that the rate of change of velocity is uniform during each half-minute. Find the acceleration in each half-minute in foot-second units. A more probable assumption would be that the rate of change of velocity diminished gradually instead of changing suddenly. How would you indicate this on your diagram?

(vi) Show as in § 18 (iv) that the acceleration, where t and v represent time and velocity, is represented by $\frac{dv}{dt}$.

(vii) A stone is thrown straight up with a velocity of 80 feet per second. The velocity decreases uniformly at the rate of 32 feet per second every second. Show this by means of a graph, and find from your diagram the time when the stone reaches its highest point, the height to which it goes, and its velocity on reaching its starting-point again. Take 10 spaces horizontally to represent 1 second, one vertical space representing a velocity of 1 foot per second.

§ **21. Value of** g. For moderate speeds, and for bodies which do not offer a large surface for air-resistance in comparison with their mass, the acceleration of gravity may be taken as uniform. The following table gives a few values of g at different latitudes :—

Latitude.	*Value of* g.	
	Foot-second units.	Centimetre-second units.
0° (Equator)	32·091	978·10
48° 50′ (Paris)	32·183	980·94
51° 29′ (London)	32·191	981·17
55° 57′ (Edinburgh)	32·203	981·54

For most purposes it will be sufficiently exact to take g as 32 or 981.

§ **22. Exercises.** (i) Fig. 17 represents the velocity curve of a stone thrown vertically downwards with a velocity of 100 feet per second. g is taken as 32. Write down from the figure (a) the speed after 4 seconds, (b) the average speed during those 4 seconds, (c) the space described during those 4 seconds.

(ii) Fig. 18 represents the movement of a point the velocity of which increases from u feet per second to v feet per second in t seconds. Show from the figure the following facts :—

$$(a)\ a\ \text{(acceleration)} = \frac{v-u}{t}\ \text{(foot-second units)}$$

(*b*) average velocity $= \frac{v+u}{2}$ (feet per second).

(*c*) s (space described in t seconds) $= ut + \frac{1}{2}at^2$ (feet).

Also obtain the following equation connecting v, u and s:—

(*d*) $$v^2 - u^2 = 2as.$$

The equations (*a*) to (*d*) are true for any set of units that correspond with one another.

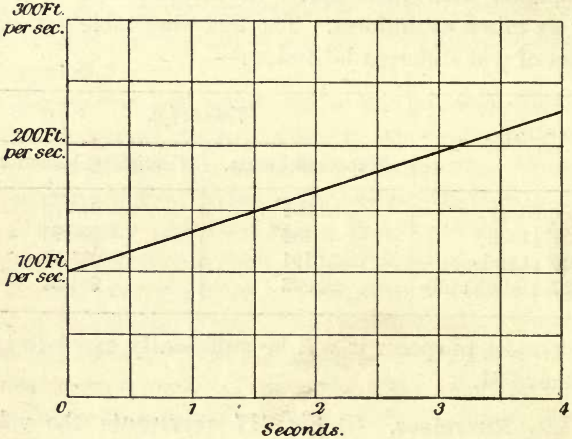

FIG. 17.

(iii) The case of a stone thrown vertically upwards with a velocity u may be regarded as a particular case of the preceding exercise by writing $-g$ for a. It is, however, of such importance that the student should draw a figure corresponding to Fig. 18, and obtain from it the formulæ

$$v = u - gt$$
$$s = ut - \frac{1}{2}gt^2$$
$$v^2 = u^2 - 2gs.$$

(iv) Show that a stone thrown vertically upwards takes the same time to come down as to go up, and that it returns to the hand with the same speed as that with which it left it. Is the velocity the same?

(v) If a shot-gun is fired straight up in the air, the pellets return harmlessly to the ground. Why is this? Is the same true for a rifle bullet?

(vi) Assuming that

$$s = ut + \tfrac{1}{2}at^2, \text{ find } \frac{ds}{dt}; \text{ also } \frac{dv}{dt}. \quad (\S\ 18\ (\text{iv}).)$$

(vii) A body starting from rest moves with a uniform acceleration of 32 foot-second units. Find how far it will move in the first, second, third, and fourth seconds respectively.

(viii) A body is projected upwards with a velocity of 200 feet per second. Find the height to which it will rise, and the time it takes to arrive at its highest position.

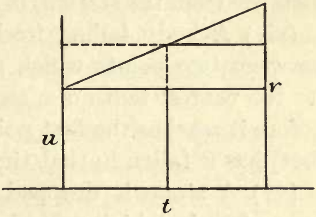

Fig. 18.

(ix) A body is projected vertically upwards with a velocity of 80 feet per second. Find and represent in a figure its positions after 1, 2, 3, 4, and 5 seconds respectively, and when at its greatest height.

If a second body is projected at the same instant with a velocity of 90 feet per second, how far apart will the two bodies be after 4 seconds?

(x) A wagon, starting from rest, runs with uniformly increasing speed down an incline. If it goes 100 yards in the first half-minute, how long will it take to run half a mile from the starting-point?

(xi) Assume the telegraph poles of a railway to be

50 yards apart. A train is observed to pass a post every 4 seconds. Calculate its speed in miles per hour. After an interval of 5 seconds it is observed to pass a post every 3 seconds. What is then the speed? Calculate in feet per second per second the acceleration (supposed uniform) during the interval of 5 seconds.

(xii) A body is suspended by a fine string at a distance 20 feet vertically above the ground. The string is cut. Find the time taken by the body to reach the ground.

(xiii) A train travelling 30 miles an hour is brought uniformly to rest at a station in $1\frac{1}{2}$ minutes. At what distance from the station, in miles, were the brakes applied?

(xiv) A body falling freely is timed to take 2 seconds between two points which are 350 feet apart. How long (to the nearest tenth of a second) has the body been falling before it reaches the first point, and how far (to the nearest foot) has it fallen in that time?

(xv) A stone is dropped from rest from the top of a tower 200 feet high. Neglecting the frictional resistances of the air, find how long it takes to reach the ground, and calculate its velocity just before it touches the ground. Plot a curve showing the velocity of the stone at various heights.

(xvi) A balloon is rising up vertically with a velocity of 60 cms. per second when a stone is dropped from the car. Describe the motion of the stone.

It is found that the stone strikes the ground 6 seconds after being dropped. Find the height of the balloon at the instant of dropping the stone.

(xvii) Find the initial velocity and the acceleration of a body which describes 30 cms. in the first 5 seconds, and 50 cms. in the next 3 seconds of its motion, the acceleration being uniform.

(xviii) Distinguish between *acceleration* and *velocity.*

The velocity of a moving body increases by 10 miles an hour in every minute. Find the acceleration in terms of yards and minutes.

(xix) How would the measure of the acceleration due to gravity be altered if the unit of time were half a second ?

(xx) Find the acceleration (assumed to be uniform) of a railway train which is brought to rest by the brakes in half a minute, when running at a speed of 20 miles per hour.

(xxi) Find the acceleration, assumed constant, of a train, that decreases in speed from 60 miles an hour to 30 miles an hour in a quarter of a mile. How much farther would the train travel before coming to rest if the retarding forces acting upon it continued the same?

(xxii) A body falls freely from rest and reaches the ground with a velocity of 40 feet per second. Find the velocity with which it would reach the ground if, for 9 feet after passing the mid-point of its descent, the body were subject to no acceleration.

Would it be possible to reproduce approximately, in an experiment, conditions similar to those described above?

(xxiii) A railway train is found to have a velocity of 30 miles per hour 3 minutes after starting. Assuming that the acceleration has been constant, find how far the train has travelled during the 3 minutes.

(xxiv) A body moves from rest over a kilometre subject to an acceleration of 5 cm.-second units. The acceleration then ceases and the body becomes subject to a retardation of 5 cm.-second units. Find its velocity after it has travelled 2 kilometres, and also the time it takes to move over each of the kilometres.

(xxv) Two particles move from the same point A along the same line AB, one of them having a uniform velocity u, and the other a uniform acceleration f and no initial velocity. Find the time that elapses before the second overtakes the first.

(xxvi) The position of a point P on a straight line is given by $x = t^2 - 2t + 3$, where t denotes the number of seconds after a fixed instant of time, and x the distance in feet of P from a fixed point O on the line. What are the positions of P at the fixed instant, and at intervals of 1 and 3 seconds after the fixed instant?

What space does P traverse between time 3 seconds and time 6 seconds, and between times 3 seconds and 3·001 seconds, and what is the average speed for each of these intervals? What is the actual speed at time 3 seconds?

(xxvii) A stone is dropped from a vertical height of 10 metres; what is its velocity on reaching the ground?

(xxviii) Supposing the resistance of the air on a falling raindrop to be proportional to the square of its speed, show that for a certain speed the acceleration is zero, so that a drop that has attained this speed will continue to fall with constant speed.

If the resistance of the air lessens the downward acceleration of the drop by 2 foot-second units when the speed is 10 feet per second, what is the constant speed?

CHAPTER III[1]

MOTION STUDIED EXPERIMENTALLY

§ **23. Laws of Falling Bodies.** These were discovered by Galileo at the end of the sixteenth century. The importance of the discovery and the wonderful combination of experimental and mathematical power required for it have been overshadowed by the subsequent work of Isaac Newton, whose Laws of Motion state in the clearest language the results obtained by Galileo. Moreover, Newton had the imagination and the power to *discover* that the range of these laws extended to the sun, moon, and planets ; and astronomers regard the movements of these bodies as affording ample proof of the truth of the laws. But the proof is only satisfactory to those who can follow the mathematical reasoning of Newton's *Principia*, and observe for themselves the movements of the planets in the stars. Any course of experimental mechanics must owe something to Galileo ; and since Galileo's apparatus was of the simplest kind, it may be useful to follow him more closely than has been customary. His experiments were made at Pisa in the years 1589–1591 A.D., but his *Discorsi e Dimostrationi matematiche*, in which he sets forth his course of experiment and reasoning, were first published in 1638[2].

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[1] This chapter may be omitted on a first reading.

[2] The first edition was printed by the Elzevirs at Leyden. I have used the copy in the Eton College Library, which belongs to the edition published at Bologna, 1655, thirteen years after Galileo's death, 'con licenza de' Superiori.'

§ **24. All bodies fall at the same rate.** For this it is important to arrange that the resistance of the air does not affect the result. The resistance of the air increases with the speed of the moving body, and depends on the area of surface which it offers. The air's action is independent of the material behind the surface, and a body of small mass will be more impeded than one of the same surface area, but larger mass. The reason for this will be entered into later.

Take a ball of lead and another of wood of about the same size. The diameters may be anything up to an inch. Ascertain their weights, which may be as 20 to 1, or 100 to 1. Hold them in one hand and drop them simultaneously from any convenient height. Do they strike the ground simultaneously ?

Galileo's experiment before the University of Pisa was performed with two leaden weights, ratio 100 : 1, dropped from the Leaning Tower. Before Galileo the authority on falling bodies was Aristotle (384 B. C.). Aristotle's treatise stated that bodies fell at rates proportional to their weights. We may quote from Galileo's comments on the objections made against his conclusions by Aristotle's followers : ' Signor Simplicio, I would not have you do as certain others do, who try to avoid the main issue by attacking a statement of mine which misses the truth by a hair's-breadth, and under this hair would hide another's mistake as big as a ship's cable. Aristotle says that a hundred-pound weight falling from a height of a hundred cubits will strike the ground before a one-pound weight has fallen more than a single cubit. I say that they will fall together. You find on trial that the larger weight reaches the ground two inches ahead of the smaller ; and under these two inches you would conceal Aristotle's ninety-nine cubits, and by speaking only of my very small discrepancy you would pass over Aristotle's very big one.'

Cut out a piece of thin paper a little smaller than a coin, hold it and the coin horizontally, and drop them together. Then lay the paper on the coin and drop it. What difference do you notice in the two cases, and what inference do you draw ?

If an air-pump and a suitable receiver are available, the ' guinea and feather ' experiment should be tried.

§ **25. The velocity of a falling body is a continually increasing quantity, starting from zero (the state of rest) and passing through all intermediate values.** In Galileo's dialogues, Sagredo objects that this statement is contrary to our experience. ' Suppose,' he says, ' that the rate of increase of speed is such that after four beats of the pulse the moving body has a speed which if maintained would carry it two miles in an hour ; then in two pulse-beats it would have a speed of one mile an hour ; then we must admit that at instants of time nearer and nearer to the first instant of its movement from rest its speed would not be enough to carry it a mile in an hour, or in a day, or in a year, or in a thousand years, or even a hand's-breadth in a still longer time : whereas our senses tell us that a heavy body gets up a great speed all at once.'

Salviati replies to this : ' You say that experience shows that a falling body acquires a noticeable speed all of a sudden ; and I say that this same experience shows that the first movements of even the heaviest bodies in falling from rest are very slow indeed. Take a heavy body and place it on some yielding material, letting it press simply with its own weight. It is plain that if we raise it a yard, or two yards, and let it fall on the same material, it will cause a new and deeper impression, due to the speed of the movement ; and this impression will be deeper and deeper as the height is increased, that is, as the speed of the concussion is increased, so that we can judge of this speed by the nature of the impression made. Now suppose we reduce the height of the fall to a hand's-breadth or an inch or the thickness of a leaf, the effect of the fall becomes more and more imperceptible, and who can doubt that the velocity, of which the mark made in the material is a sign, also becomes so small as to be imperceptible ? ' (*Dialogue* III).

This experiment can be made with a small metal ball on a flat piece of putty, or with a heavy weight, such as a 16lb. shot, on a smooth piece of soft ground.

§ **26. If the velocity of a falling body increases continually, according to what law does it increase ?** Here Galileo has recourse to argument. He says, ' Bodies will fall in the most simple way, because natural motions are always the most simple. Nature always employs the simplest and easiest methods. Nobody would believe that swimming or flying could be done by simpler and easier processes than those used by fishes and birds. Why not then suppose that the increase of speed of a falling stone follows the most obvious law, and what law more simple than that in which the increase of speed is proportional to the time ? '

Objections may be made to this kind of argument. 1900 years before this, Aristotle had argued, and had landed himself in the statement that heavy bodies fall more quickly in proportion to their weight. But he can never have tested this conclusion by experiment. Galileo, on the other hand, made experiments and found that they confirmed his assumption. It is the way in which Nature's laws are discovered. Kepler made many wrong assumptions for the laws of planetary motion, and found that none of them suited the facts. At last he hit upon an assumption that did fit in with the results of Tycho Brahe's observations, and in this way he discovered his laws. Galileo appears to have discovered his law for the increase of speed of a falling body at the second attempt. He had previously assumed the velocity to be proportional to the space travelled, instead of the time of falling.

The direct experimental proof that the velocity is proportional to the time is not easy. It is easier to measure the space fallen in a given time than the speed at the end of the time. Also the speed of bodies falling freely increases so rapidly that the measurement of time is a difficulty. It is not easy to time a hundred-yards race correctly to a fifth of a second, and a stone that has been falling for one second is moving faster than the fastest hundred-yards runner. We may with advantage follow Galileo closely here. His method was :

(i) to calculate the space fallen on the assumption that the speed was proportional to the time ;

(ii) to ' dilute ' the force of gravity, and study the motion of a smooth metal ball down a polished groove in a long inclined plane of gentle slope.

§ 27. Graphical representation of uniform speed. Taking two straight lines at right angles, on squared paper preferably, we will mark off equal spaces along one of these lines to represent equal times (Fig. 19).

Along the other line we mark off speeds. The straight line *PN* obviously represents a uniform speed of 2 ft. per second, and the area *PNOM* is six squares, and may be taken to represent the space travelled in 3 seconds by a body moving at the rate of 2 ft. per second. The shaded area represents the space travelled in the third half-second. We may, of course, choose our units differently.

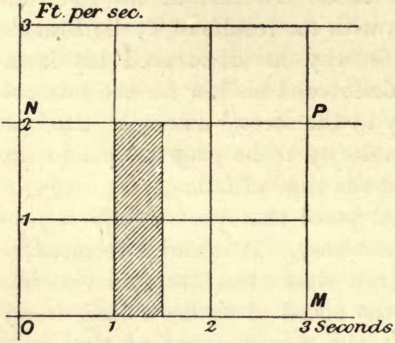

Fig. 19.

(i) In Fig. 20 each square represents the space travelled in five seconds by a body moving five miles an hour. What is this space? What does the shaded area represent?

(ii) Show on squared paper the distances travelled by a train going 20 miles an hour for 5 minutes, and another going 35 miles an hour for $2\frac{1}{2}$ minutes, and compare them.

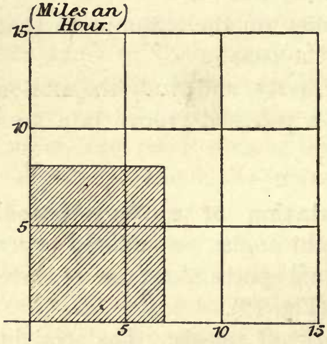

Fig. 20.

(iii) What is the distance travelled in y seconds by a train going x miles an hour? Give the answer in feet.

(iv) Let s represent the number of feet travelled in t seconds by a body moving at the rate of v feet per second. What equation connects the three letters s, v, t? How would the equation be altered if v stood for the number of miles per hour instead of feet per second?

(v) Express one mile an hour in centimetres per second.

We see then that the space travelled is proportional to the product of the velocity and the time; and if the units are chosen so as to agree with one another, e. g. if space is measured in centimetres, time in seconds, and velocity in centimetres per second, the space is equal to this product, i. e. $s = vt$ (Galileo, *De Motu Æquabili*, Prop. iv).

§ **28. The distance travelled from rest in any given time by a body moving with uniformly increasing speed is the same as if the body moved for the same length of time with a uniform speed equal to half the final speed attained in the given time.** Let OM (Fig. 21) represent the time, MP the velocity at the end of the time. Join OP by a straight line. Then it is plain that the height of this straight line above OM increases uniformly as we go from O to M; and the velocity at any time represented by OR is itself represented by the perpendicular QR. Now bisect PM in W, and draw WTV parallel to OM. It is plain that the parallelogram $OVWM$ is equal to the triangle OPM. The small area contained between QR and $Q'R'$ represents

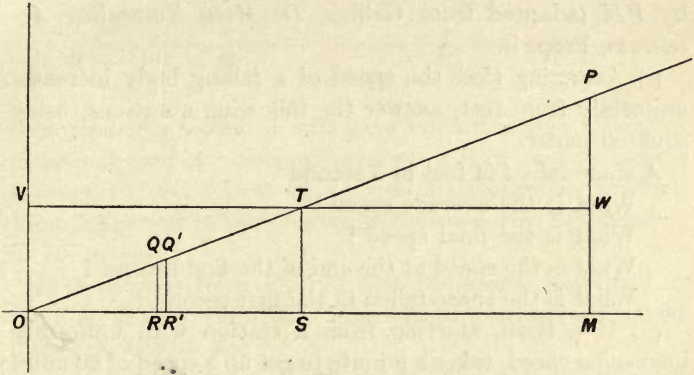

FIG. 21.

the space travelled in the small interval of time represented by RR', for it is intermediate between the two parallelograms on the base RR' and with heights QR, $Q'R'$. (See § 19.) The triangle OPM may be considered as built up of strips such as this, so that on the same scale the area OPM represents the area travelled with the uniformly increasing velocity. Meanwhile it is obvious that the parallelogram $OVWM$ represents the space travelled in the time OM by a body moving with a uniform average speed equal to half the final speed represented by PM (adapted from Galileo, *De Motu Naturaliter Accelerato,* Prop. i).

(i) Assuming that the speed of a falling body increases uniformly from rest, answer the following questions, using squared paper.

A stone falls 144 feet in 3 seconds.

What is the average speed?
What is the final speed?
What is the speed at the end of the first second?
What is the space fallen in the first second?

(ii) If a train, starting from a station with uniformly increasing speed, takes a minute to get up a speed of 60 miles an hour, how far does it go in the first minute?

§ 29. The spaces travelled by a body moving with uniformly increasing speed in any times whatsoever are to one another as the squares of the times (reckoned from the beginning of the motion) (Galileo, *De Mot. Acc.* Prop. ii). Let OM, OM' (Fig. 22) represent the two times measured from O, the starting-point. Then if PM, $P'M'$ represent the velocities at the ends of those times, OPP' is a straight line; for the velocity increases uniformly from rest.

Now the distances travelled in those times are represented by the areas OPM, $OP'M'$, which are proportional to the squares on OM, OM'.

Hence supposing that a body falls with uniformly increasing speed through one foot in a quarter of a second,

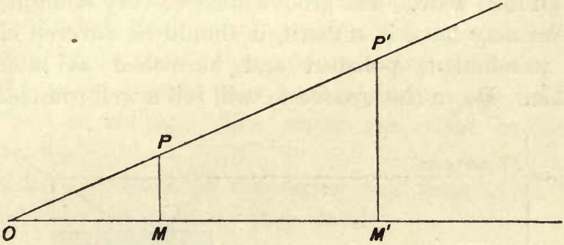

FIG. 22.

then in half a second it will have fallen through 4 feet, in three-quarters of a second through 9 feet, in one second through 16 feet. At the same rate of acceleration, through what space will it have fallen in 3 seconds? in $5\frac{1}{4}$ seconds? in 10 seconds?

It is obvious from Fig. 23 that the spaces described in the first, second, third, &c., intervals of time are as the odd numbers, 1, 3, 5, &c.

(i) If a stone falls 16 feet in the first second, how far (making the same assumption as before) does it fall in the 10th second? How far in 10 seconds? How far in n seconds? How far in the nth second?

(ii) What is the sum of 1, 3, 5, &c., to 10 terms? to n terms?

§ 30. Experimental proof that the space travelled under the action of gravity is proportional to the square of the time. Galileo made use of a long inclined plane with a straight polished groove in it, down which he allowed a smooth ball to roll.

We may as well give his own account of his apparatus. ‘In a wooden rafter about twelve cubits long, about half a cubit in depth, and three inches in breadth let there be gouged out along the narrower side a groove a little more than an inch wide. The groove must be very straight, and, that we may have it smooth, it should be covered closely with parchment, polished and burnished as much as possible. Down this groove we will roll a well-rounded and polished ball of hard bronze, the rafter being inclined by raising one of its ends above the horizontal plane, one, or two cubits, as we please ; and the time taken by the ball in running down the full length of the groove must be noted in a way which I will explain afterwards. This must be repeated many times, to ensure from the number of observations that there should be no difference in the results, not even as much as the tenth part of a pulse-beat. The result having been established, let the same ball roll down only one quarter of the groove ; its time of descent will be

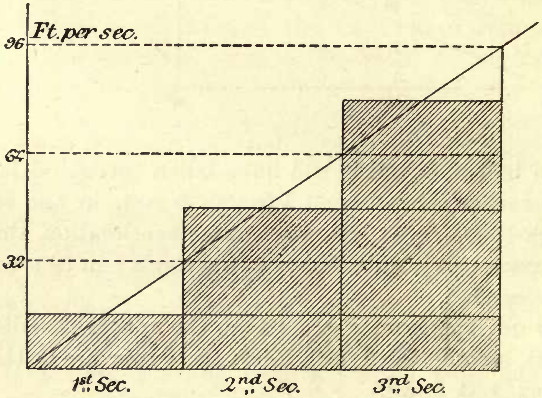

Fig. 23.

found to be exactly half the other time. Repeating the experiment with other fractions of the length it will be found that the spaces travelled are in every instance as the squares of the times. . . .

Now for the method of measuring the time. Have a large bucket of water on a shelf, which, by a narrow tube soldered on to the bottom, pours a fine thread of water which can be caught in a beaker during the time that the ball is rolling. The water can then be weighed accurately.'

Galileo's method of measuring the time gives better results than the ordinary stop-watch.

§ 31. A simple adaptation of Galileo's experiment. Satisfactory results can be obtained by using a piece of hard wood for the groove. A piece of mahogany, or oak, will take sufficient polish without the parchment covering, and motor-car bearings can be easily obtained from any cycle agent. The following method of verifying the law of the squares of the times does not involve the use of any instrument for measuring time, and works perfectly well with an inclined plane only four feet long.

Divide the plane *AB* (Fig. 24) accurately into four. Procure two steel balls of the same size. Let one start from *B*, and have the other ready at *C* so that it can be set free at the instant that the first ball passes the point *E*. The

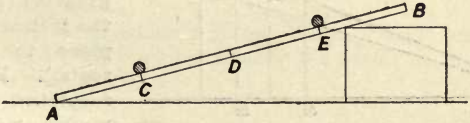

Fig. 24.

two balls will reach A simultaneously. This can be repeated with the plane at different inclinations.

§ 32. Experimental proof that the speed at the bottom of the incline is double the average speed of the descent. For this it is necessary to have a horizontal continuation of the groove. If the slope of the incline is gentle, it is easy to adjust the two pieces of wood so that the ball rolls on to the flat without interruption. The horizontal part must be carefully levelled, and should be tested for smoothness by rolling the ball very slowly along it either way (Fig. 25).

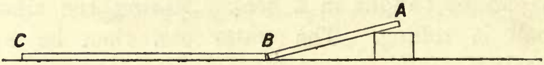

Fig. 25.

Mark off BC double the length of AB. Let a ball start from A, and the instant that it passes B let another ball start from A. The second ball should reach B as the first reaches C. Can you account for any discrepancy you may observe? Explain the bearing of this experiment on the statement of § 28.

§ 33. The space described in a given time by a ball rolling down an inclined plane is proportional to the gradient (the sine of the inclination of the plane to the horizontal). For this an adaptation of Galileo's method may be employed. Two inclined planes (AB, AC) are supported with their lower ends level with each other at A. The blocks PM, $P'M'$ are of the same height (Fig. 26).

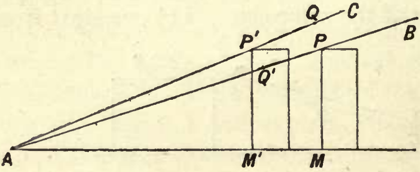

Fig. 26.

Now if the space described down AB in a given time is to the space down AC in the same time as $\frac{PM}{AP}$ is to $\frac{P'M'}{AP'}$, it follows that

$$\frac{\text{space down } AB}{\text{space down } AC} = \frac{AP'}{AP}.$$

If now we take points Q, Q', such that $AQ = AP$, and $AQ' = AP'$, and place balls at these points, letting them go simultaneously, it will be found that they reach A simultaneously. This experiment is easily made. The two planes should first be tested with the same inclinations, to make certain that the conditions of the two grooves are the same. Afterwards, the positions of the blocks PM, $P'M'$ can be varied to any extent.

This experiment should be studied in connexion with another, the measurement of the force necessary to support a weight on an inclined plane. For instructions for performing this the reader should refer to § 65, where the apparatus is described.

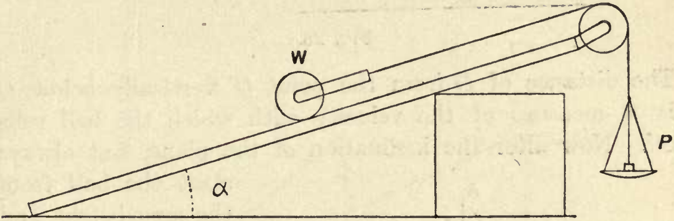

FIG. 27.

Fig. 27 indicates the method. W is a smoothly running heavy roller supported by a weight (P) hanging over a smooth pulley. At different values for the angle of inclination α it will be found that $P = W \sin \alpha$.

Hence the statical force necessary to support the weight is proportional to the gradient.

From these two experiments follows the important fact that the acceleration (see § 19) is proportional to the force. Another verification of this is given in § 45.

§ 34. To prove that velocities acquired in rolling down planes of the same height but different lengths are the same. Fig. 28 shows the arrangement. The groove AB is continued horizontally for a short distance to the edge of the table, and a small box (*D*) with cotton wool inside is placed on the floor to catch the ball when it rolls off at *C*.

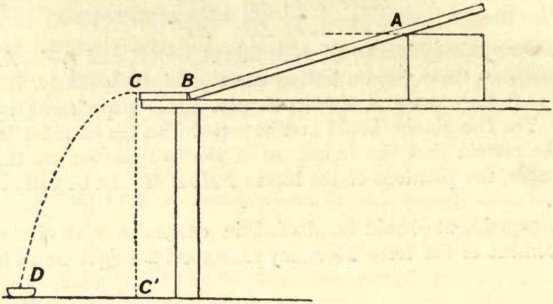

FIG. 28.

The distance of *D* from the point *C′* vertically below *C* is a measure of the velocity with which the ball rolls off. Now alter the inclination of the plane, but always start the ball from the same height, and it will be found that the position of *D* needs no alteration.

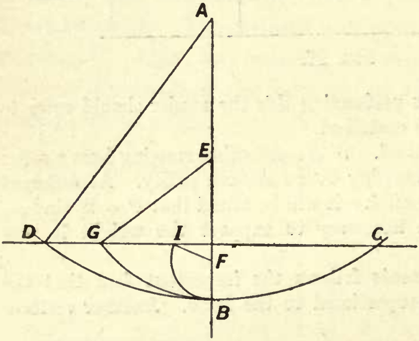

FIG. 29.

Galileo's experimental proof was as follows. A ball (*B*, Fig. 29) is suspended by a fine string from a nail (*A*) and a horizontal line (*CD*) is drawn on the wall below *A*. Now the ball is pulled back to the point *C* and let go. The speed it acquires at *B* is enough to carry it very nearly up to the line at *D*, the resistance of the air and the thread stopping it very slightly. Now put another nail at *E* or *F* vertically below *A*, so as to catch the thread when the ball is at its lowest point. Then *E* or *F* will become the centre of the arc described, and the velocity of the ball will be enough to carry it nearly to *G* or *I* in the same horizontal line *CD*, showing that the velocity destroyed in ascending the arcs *BI*, *BG*, *BD*, or acquired in descending them, are all equal. From this he assumes the same thing to be true of inclined planes of different lengths but the same height.

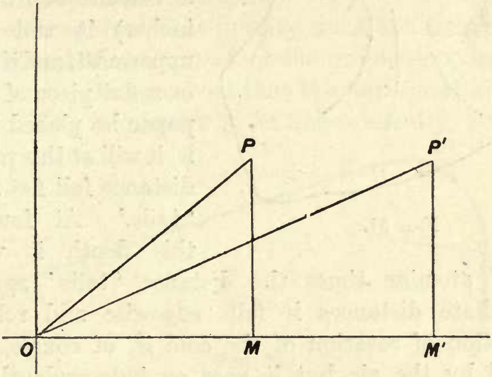

FIG. 30.

It is obvious from Fig. 30 that if *POM*, *P′OM′* represent the spaces described from rest in the times *OM* and *OM′*, and if

$$\frac{\Delta^{e}\,OPM}{\Delta^{e}\,OP'M'}=\frac{OM}{OM'},\ \text{then } PM=P'M';$$

$$\text{for } \Delta^{e}\,OPM=\text{half } OM\times PM$$

$$\text{and } \Delta^{e}\,OP'M'=\text{half } OM'\times P'M'.$$

But *PM*, *P′M′* represent the final velocities. Therefore these final velocities are equal.

§ **35. Another experimental verification** of the law that the distance fallen varies as the square of the time is due to Mr. G. F. C. Searle. Hold a coin with the finger and thumb as in Fig. 31. When the finger is drawn away at *A* the coin falls, and the pressure of the thumb at *B* gives it a turning movement which goes on uniformly during the fall. Supposing the coin to be heads uppermost to start with. At a certain depth (a few inches) it will be tails uppermost, and if the hand or a flat piece of blotting paper be placed to catch it, it will at this particular distance fall flat and true 'tails.' At four times this depth it will fall 'heads,' at nine times the distance 'tails' again. At intermediate distances it falls edgewise and rolls over. The motion of rotation of the coin is, of course, slightly retarded by the air, but it goes on independently of the falling movement: and this brings us to the question of a body possessing at the same time two different movements.

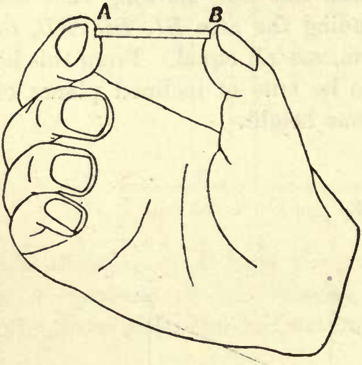

FIG. 31.

§ **36. Motion of a body projected horizontally.** It may be useful here to point out some of the old ideas held before Galileo's time as to what happens to a cannon-ball fired horizontally. Santbach in 1561 maintained that it went in a straight line till all its velocity was exhausted and then fell straight down, gravity acting alternately with the original force of projection (see Fig. 32). On this assumption he based a treatise on gunnery. Another assumption was that the path consisted first of a straight line, then an arc of a circle until the motion became vertical downwards, and then straight down (Fig. 33).

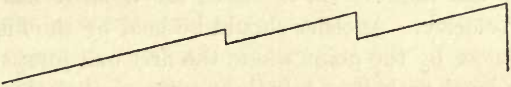

FIG. 32.

Galileo recognized that gravity must act from the very beginning and that the path of the projectile is nowhere a straight line. He perceived that the horizontal movement and the falling movement go on independently, and on the assumption that the horizontal movement continued unchanged, he found the path to be a parabola. As a matter of fact the path of a cannon-ball or a bullet is very far from being a parabola; but this is because of the resistance of the air, which is a most important factor in cases where the velocity is large; but for moderate speeds the assumption, that the horizontal velocity is unchanged, is justified by results. The student should study what follows as bearing directly on the statement of § 11, particularly as to the turning effect of a force acting across the line of motion.

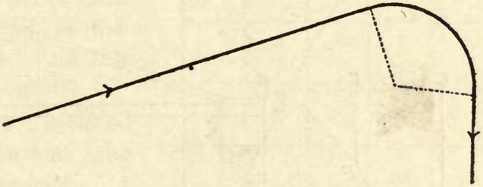

FIG. 33.

§ **37. The vertical motion is independent of the horizontal motion.** Arrange the inclined plane as in Fig. 28, and allow the ball to roll down it and off the edge of the table. As it leaves the table it has a horizontal velocity. Another should be held by the finger and thumb close by the point where the first ball jumps off and exactly level with it. A little practice will enable you to drop the second ball at the instant the first ball passes it. The two will strike the floor simultaneously, showing that the ball with the horizontal velocity falls as fast as the other.

§ **38. To find the path of a ball projected with a horizontal velocity.** For this purpose a grooved inclined plane with a horizontal continuation should be placed on a shelf close to a wall or a vertical blackboard on which lines can be marked (a squared blackboard would be suitable). The ball is started from *A* and rolls off at *B*.

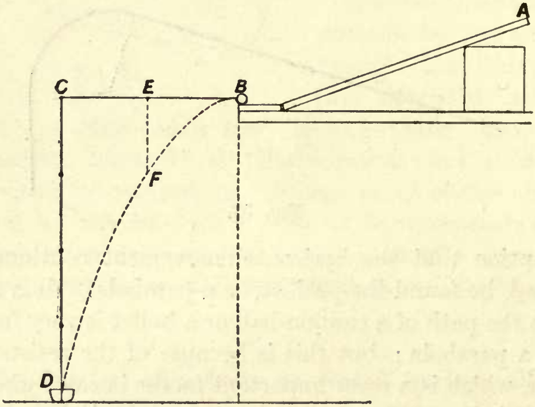

Fig. 34.

A cup *D* is placed on the ground by trial to receive the ball, and lines *BC*, *CD* are marked vertically and horizontally. Bisect *BC* at *E*, and draw *EF* vertical and equal to a quarter of *CD*. *F* should be opposite a point on the ball's path. Of course the ball must be started from the same point every time. A paper hoop or a wire ring may be fastened opposite for the ball to go through.

If the horizontal distance is proportional to the time and the vertical distance to the square of the time, it is easy to mark out other points on the curve. Fig. 35 indicates how this may be done. *BH* is one-sixth of *BC*, and *HK* is one thirty-sixth of *CD*. Similarly *BL* is one-third of *BC*, and *LM* one-ninth of *CD*.

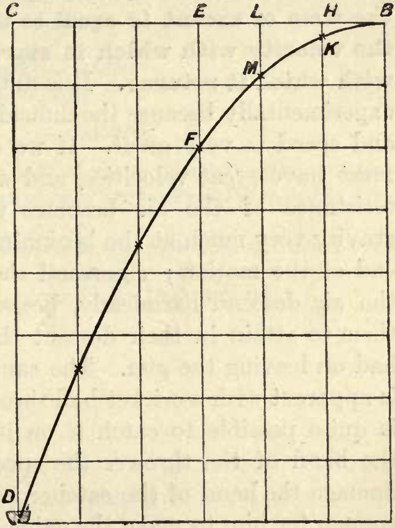

Fig. 35.

(i) An inclined plane is 10 feet long and 1 foot high. A ball rolling down this plane acquires a velocity of 8 feet per second and rolls off the table horizontally. Plot a reduced plan of the path on squared paper, taking half an inch or a centimetre as representing one foot.

What is the vertical space fallen through in one second? What is the vertical velocity at the end of one second? At the end of two seconds? three seconds? At what rate per second does the vertical velocity increase? Does this correspond with your own practical results?

(ii) Plot on squared paper the curves described by balls projected horizontally from the same point with velocities of 16, 20, 32, 40, 80 feet per second.

(iii) Plot the graphs of

$$y = x^2$$
$$y = \tfrac{1}{4}x^2, \ y = \tfrac{1}{16}x^2.$$

§ **39. When a body is projected vertically upwards, the time of ascent is equal to the time of descent, and the velocity with which it starts is the same as that with which it returns.** It is difficult to prove these facts experimentally because the difficulty of measuring the time and speed is very great. If we employ great heights we must have great velocities, and when these are used the resistance of the air becomes very noticeable, helping gravity very much at the beginning and opposing it at the end of the motion; e.g. small shot fired straight up into the air descend harmlessly, because the air never allows them to attain in their descent the same velocity as they had on leaving the gun. The same thing to a less degree is apparent with a cricket ball thrown hard straight up. It is quite possible to catch it on its return; but as it left the hand of the thrower the speed would be enough to damage the hand of the catcher, even if his eye was quick enough for him to make the catch.

§ **40. Resistance of the Air.** Galileo, who discovered that the acceleration due to gravity is practically uniform, was fully aware of the important part played by the resistance of the air when the speed becomes very great. He points out (*Dialogue I*) that a falling body moves with increasing speed, and that as the speed increases the resistance of the air increases until at length a speed is reached at which the retardation caused by the air exactly balances the acceleration of gravity, and when this happens the speed remains uniform during the subsequent motion. The speed at which this happens is different for different bodies, depending on the shape and the mass of the body as well as on the area of the surface exposed to the resistance of the air.

Suppose that the maximum velocity of falling for a given body is 1000 feet per second, and let us assume that the resistance of the air is proportional to the square of the velocity. It is then possible to ascertain the acceleration of the body at any given velocity. For if the resistance is proportional to v^2 we may take it as Kv^2, where K is some constant number. The acceleration of gravity g must be diminished by this resistance, making the net acceleration $g-kv^2$. Now g in foot-second units is 32, and v is 1000 when the velocity becomes uniform, i.e. when $\frac{dv}{dt}=0$.

This gives

$$0 = 32 - k \times 1000^2$$

$$\therefore\ k = \cdot 000032.$$

$$\therefore\ \frac{dv}{dt} = 32 - \cdot 000032 v^2.$$

Find the acceleration of this body when the velocity becomes 100 feet per second. Also find for what velocity the acceleration is half of that due to gravity alone.

In order that the velocity-time curve may be plotted, it will be necessary to *integrate* the above *differential* equation, i.e. obtain an equation connecting v and t. The required equation is $v = 1000 \tanh \cdot 032\, t$.

CHAPTER IV

MASS AND FORCE

§ **41.** A roller weighing anything over $\frac{1}{2}$ lb., which runs smoothly on an axle, and a smooth plane with a pulley, over which a light scale-pan can be hung, are wanted. Find the weight of roller and scale-pan. Let the plane be horizontal. Put the roller on it, and attach the scale-pan, as shown in Fig. 36. If the scale-pan is heavy enough to start the roller, a lighter pan should be substituted, or even a small muslin or paper bag. Ascertain the smallest weight (including the weight of the scale pan) which is necessary to start the roller.

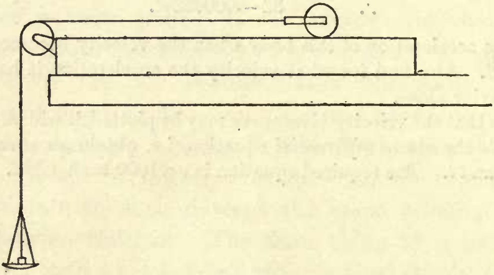

FIG. 36.

Find also what weight is necessary to keep the roller in steady motion if it is given a start by hand, i.e. what weight is necessary just to prevent its coming to rest.

Experiments of this kind should not be hurried over. It is important that the student should realize how small a force is necessary to overcome the resistance of friction, when a smooth-running roller or trolley is placed on a horizontal plane.

When this force has been ascertained, it should be noticed that any additional force produces an accelerated motion. It is necessary, however, to have some means of measuring the acceleration, and finding the relation connecting it with the force and the mass moved.

§ **42. Measurement of Acceleration.** For this purpose an apparatus, designed by Mr. W. C. Fletcher, may be employed with advantage.

AB (Fig. 37) is a trolley, consisting of a flat board, 2 to 3 feet long, and 3 inches wide, mounted on two pairs of smooth-running wheels. *P* is a paint-brush mounted on the end of a long steel rod clamped at the other end so that *P* can vibrate at right angles to the length of the trolley. A strip of drawing paper should be pinned closely over the top of the trolley, and the paint-brush adjusted so that it just touches the paper as the rod vibrates. Move the trolley by hand under the vibrating brush. A wavy curve will be traced out on the paper : Fig. 38 shows a reduced copy of one of these curves. The vibrations of the rod are made in equal times : hence it is obvious that the distances between the crests of the waves represent

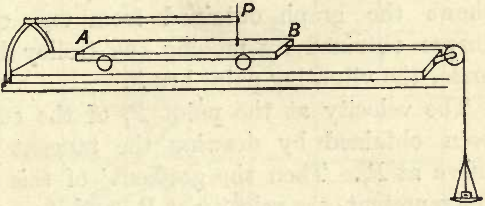

Fig. 37.

distances described in equal times by the trolley. Suppose that the rod vibrates five times per second, then in successive fifths of a second the trolley moves through distances *AB*, *BC*, *CD*, &c. If the trolley is well made, it can be run back along the same path, and the paint-brush, if at rest, will trace out the centre-line of the wave-curve. Then the wave-lengths can be measured between the points where the curve cuts this line. (See Fig. 40.)

Plot the space-time graph for one of these curves. Mark off equal distances along the axis of x to represent fifths of a second. Then take as ordinates the distances *AB*, *AC*, *AD*, &c. Fig. 39 shows the graph obtained from one of these curves obtained by moving the trolley by hand under the vibrating paint-brush.

The velocity at the point P_1 of the curve has been obtained by drawing the tangent to the curve at P_1. Then the gradient[1] of this tangent line represents the velocity at P_1, and if we produce the line so that the base B_1C_1 is one second, then the perpendicular A_1C_1 represents the space per second, i.e. the velocity at P_1. Take a point Q_1 above the same point of time as P_1, making its height equal to A_1C_1. Similarly the height of Q_2 is equal to A_2C_2, which represents on the same scale the velocity at P_2. In this way the velocity graph may be obtained. It is merely indicated in Fig. 39 by a number of points.

On examining the velocity graph we find that the

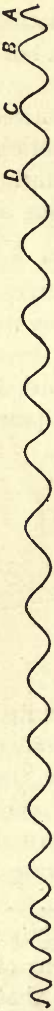

FIG. 38.

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[1] Gradient here means $\frac{\text{perpendicular}}{\text{base}}$.

velocity increases rapidly to begin with, then more slowly till it reaches a maximum, when it begins to decrease, finally diminishing rapidly.

This is exactly what has happened in the case of the force applied by the hand in moving the trolley. The impelling force was strong to begin with, and the retarding force strong to finish up with. There must have been an intermediate instant when no force at all was acting. Now if we plot the slope of the velocity-time graph, we get what we may call the acceleration-time graph. This graph crosses the axis of x at the time when the velocity graph was horizontal. Afterwards the acceleration is negative, for the tangents of the angle of slope of the

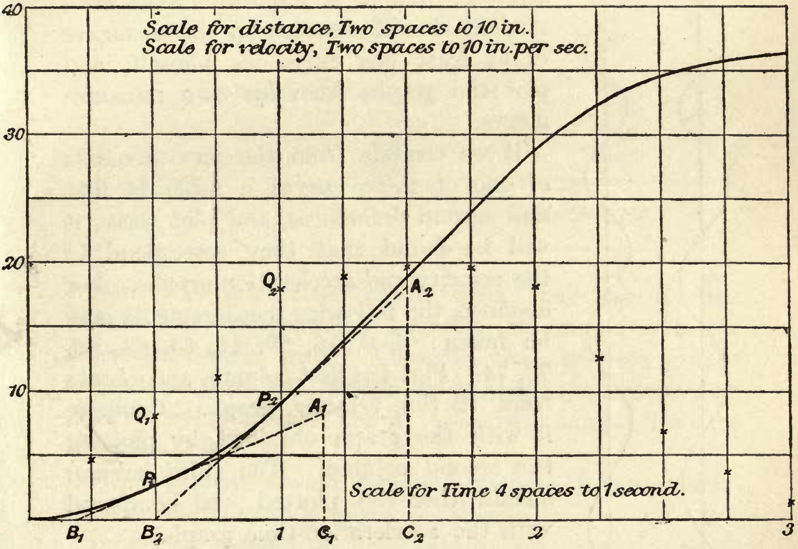

FIG. 39.

velocity graph are all negative, the perpendicular in each case being below the base.

A large number of these wave-curves can be prepared at one time. The method of preparing them can then be demonstrated: but it is important that each student should measure up the distances of at least one curve for himself, and plot the graphs from his own measurements.

If we compile from the measurements of one of these curves a table of first and second differences, and plot these, it will be found that they correspond to the velocity and acceleration graphs. For instance, the following measurements may be taken: 3, 9, 18, 29, 41, 55, 64, 69, 73, 74. Plot the first column, and obtain from it the velocity graph. Compare it with the graph obtained by plotting the second column. The third column should then be plotted and compared with the acceleration-time graph.

§ 43. To verify that the Acceleration is inversely proportional to the Mass, if the force is constant. Let the trolley be drawn by a weight passing over a smooth pulley, as shown in Fig. 37. A facsimile of two of these curves is shown in Fig. 40. Observe the steadily increasing length of the waves. The

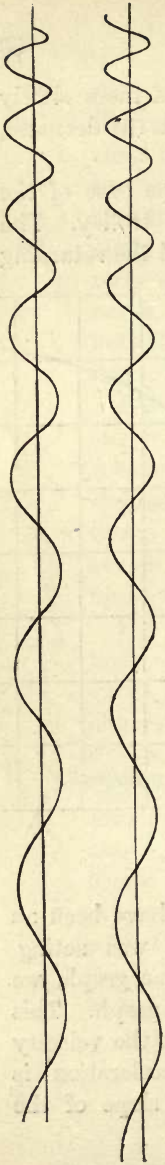

FIG. 40.

measurements obtained from one such wave-curve were as follows :—

Distances measured from *A*.	1st Differences, i.e. wave-lengths.	2nd Differences, i.e. increase in wave-length.
0·65		
1·55	0·9	
2·85	1·3	0·4
4·55	1·7	0·4
6·65	2·1	0·4
9·2	2·55	0·45
12·1	2·9	0·45
15·45	3·35	0·45
		Average 0·425

Plot the space-time graph from the first column. Obtain the corresponding velocity-time graph, and the acceleration-time graph. Compare the last two graphs with those that you obtain by plotting the second and third columns in the above table with equal times as abscissae (§ 30 (ii)).

Each student should measure up one of these wave-curves, and plot the graphs from them.

It will be noticed that the velocity-time graph is practically a straight line, i.e. that the velocity increases uniformly, and that the acceleration graph is horizontal, which is merely another way of showing the same fact. We can now save the labour of plotting the graphs, and obtain the acceleration by taking the differences in the wave-lengths of the curve.

It is now necessary to compare the accelerations obtained when the same force acts on different masses. Here we propose to quote Mr. Fletcher (*School World*, May, 1904).

‘Our trolley weighs about 400 grammes: a 50-gramme weight gives suitable results.

‘Next, the load on the trolley is varied, and the accelerations produced by the same falling weight observed. To get satisfactory results it is necessary first to find the weight which is required in each case just to maintain motion[1]. To illustrate I give a set of actual results:—

I.

Trolley	407 grs.
Pulling weight	50 ,,
Resistance balanced by	4 ,,
Total mass moved	461 ,,

Distance.	Speed.	Acceleration.
0	—	—
35	35	—
82·5	47·5	12·5
143	60·5	12·5
214·2	71·2	10·7
298·3	84·1	12·9
394	95·7	11·6
503	109	13·3
	Average	12·4

II.

Trolley + load	807 grs.
Pulling weight	50 ,,
To balance resistance	8 ,,
Total mass moved	865 ,,

Distance.	Speed.	Acceleration.
0	—	—
17	17	7·3
41·3	24·3	19·7÷3
—	—	
110	—	
154	44	
205	51	7
262	57	6
325·2	63·2	6·2
395·7	70·5	7·3
473·2	77·5	7·0
	Average	6·7

Comparison of results $\frac{461}{865} = 0{\cdot}535$. $\frac{6{\cdot}7}{12{\cdot}4} = 0{\cdot}540$.

i.e. the acceleration is inversely proportional to the mass.’

§ 44. Tension of string moving uniformly. Effect of Change of Speed. The trolley referred to in § 42 can be utilized in showing that the pull on a moving weight is the same as when the weight is at rest if the motion is uniform; but that it alters when the speed is changing. Fig. 41 shows the arrangement. Find out what the spring-balance registers when the scale-pan hangs from it at rest. Then adjust the slope of the plane so that the trolley will pull the weight up with uniform speed when started by hand. Read the balance when the speed is uniform. Then tilt the plane to a steeper angle, so that the trolley pulls the weight up with increasing speed. Is there any difference in the balance reading? (For this purpose a spring-balance reading single grammes up to 100 is suitable.)

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[1] Note that the uniformity of the motion should be tested by the vibrating spring. If the speed is uniform the wave-lengths will be equal.

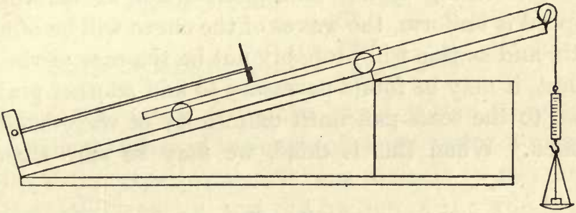

FIG. 41.

§ 45. The acceleration of a moving body is proportional to the force acting on a body. This can be verified experimentally by means of the trolley. The variable force is obtained by allowing the trolley to roll down an inclined plane, the slope of which can be varied. To avoid having to consider the force of friction, the trolley should first be connected by a string, passing over a pulley, to a scale-pan of known mass, and weights should be added just sufficient to prevent the trolley from starting to roll down the plane. It is important to find this correctly, as it gives us the measure of the force actually causing the acceleration, down a plane of this particular slope or gradient, which is produced when the string is detached.

Fig. 42 shows the arrangement, adapted from Mr. Fletcher's apparatus. When the scale-pan and weights have been adjusted roughly by trial, the adjustment should be tested by starting the trolley moving down the plane, and finding whether its speed is uniform or not. This can be tested by the vibrating spring and paint-brush as before. If the speed is uniform, the waves of the curve will be of equal length, and as this will probably not be the case at the first attempt, it may be found necessary to add another gramme or two to the scale-pan until uniformity of wave-length is obtained. When this is done, we may be sure that the total force acting on the trolley down the plane is balanced by the scale-pan and weights, and the friction of the pulley, which should be very small compared with the other forces. Now if the string is removed the trolley will roll down the plane under the action of this same force. The wave curve will now show acceleration which can be measured as before. The following is an actual set of measurements :—

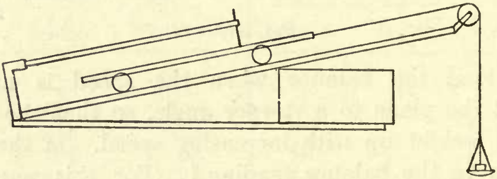

Fig. 42.

Weight of scale-pan, 21 grammes.

First observation : with 8 grammes in the scale-pan the trolley was found to move with uniform speed when started down the plane.

∴ when the string was detached the force down the plane was equal to the weight of 29 grammes. Wave-lengths 1·7, 2·6, 3·6, 4·55, 5·5, 6·5, 7·4, 8·3. Average difference, 0·95.

Second observation (with plane at a different slope): scale-pan + 30 grammes = 51 grammes. Wave-lengths, 2·25, 3·85, 5·55, 7·2, 9, 10·6. Average difference, 1·67.

$$\text{Ratio of forces} = \frac{51}{29} = 1{\cdot}76\ldots$$

$$\text{Ratio of accelerations} = \frac{1{\cdot}67}{0{\cdot}95} = 1{\cdot}76\ldots$$

Hence the ratios were the same within 1 per cent. The gradients (sines of the angle of slope) were not measured in the above experiment. It was difficult to measure the gradient very exactly, and the friction of the wheels of the trolley was considerable, so that the force down the plane was not proportional to the gradient. The success of the experiment does not depend on friction being got rid of (except in the case of the pulley), but on the accurate measurement of the force necessary to balance the total force down the plane.

§ **46.** We have therefore proved experimentally the following facts :—

(i) When the same force acts on different masses the acceleration produced is inversely proportional to the mass.

From this it must follow that, if the mass is increased, the force acting on it must also be increased in the same ratio, if the acceleration is to be unaltered.

This is the case with the Earth's attraction. The weight of a body is proportional to its mass, the pull of the Earth increasing with the 'quantity of matter,' just as we employ ponies for light loads and cart-horses for heavy loads.

(ii) When the same mass is acted on by different forces, the acceleration is proportional to the force.

Putting these facts into the language of algebra, we take F, m, a to represent force, mass, acceleration.

Then $a \propto \frac{1}{m}$ when F is constant, and $a \propto F$ when m is constant.

Hence when both F and m vary

$$a \propto \frac{F}{m} \quad \text{or} \quad F \propto ma.$$

$\therefore$ $F = kma$, where k is a constant quantity, depending only on the units employed.

The constant may be found approximately for a set of measurements with the apparatus described in § 45. The moment of inertia of the trolley-wheels should be negligible (see § 106). For instance,

time of vibration of spring = $\frac{1}{5}$ sec.;
increase of wave-length = 1·56 cm.;
weight of trolley = 1 kilogramme;
weight (including scale-pan) necessary to make the velocity down the incline uniform = 40 grammes.

In every $\frac{1}{5}$ sec. the distance travelled in $\frac{1}{5}$ sec. increases by 1·56 cm.

In every $\frac{1}{5}$ sec. the velocity increases by $5 \times 1{\cdot}56$ cm. per sec.

In every second the velocity increases by $5 \times 5 \times 1{\cdot}56$ cm. per sec.

$\therefore$ acceleration (in cm.-sec. units) $= 25 \times 1{\cdot}56 = 39$.

$$F = kma.$$
$$F = 40 \text{ grammes weight.}$$
$$m = 1000 \text{ grammes.}$$
$$a = 39.$$

Hence when the force is measured in grammes weight, the mass in grammes, and the acceleration in cm.-sec. units, we have

$$40 = k \times 1000 \times 39.$$

$$\therefore \quad k = \frac{1}{975}.$$

Now, owing to experimental inaccuracies[1], this value of k is not quite correct. It should be $\frac{1}{981}$ in the latitude of London, and at the Equator $\frac{1}{978}$. The difference is due to the fact that the weight of a gramme is different in different places (see § 6). Assuming that in our latitude $k = \frac{1}{981}$, it will be convenient to take as our unit of force $\frac{1}{981}$ of the weight of a gramme. To reduce our force to these units, we must multiply the weight in grammes by 981. We have thus

force in grammes weight × 981 = mass in grammes × acceleration in cm.-sec. units.

The above conclusions are stated by Newton in his Second Law of Motion. Taking the word momentum to signify mass × velocity, this law may be stated thus :—

Change of momentum (or rate of change of momentum) is proportional to the force, and takes place in the direction of that force.

§ **47. Units of force. A force that will produce in a mass of 1 gramme an acceleration of 1 cm. per sec. per sec. is called a dyne.** A dyne is about $\frac{1}{981}$ of the weight of a gramme in the latitude of London, $\frac{1}{978}$ of the weight of a gramme at the Equator. Roughly, we may regard a dyne as about equal to the weight of a milligramme.

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[1] The error is usually more than in the instance given.

Imagine a pound and a gramme to be dropped simultaneously. They fall with the same acceleration (981 cm.-sec. units). The pound is acted on by its own weight, which, although greater than that of the gramme, is acting on a correspondingly greater mass. Now 1 ft. = 30·48 cm.

$$\therefore \quad 981 \text{ cm.} = \frac{981}{30{\cdot}48} \text{ ft.} = 32{\cdot}18 \text{ ft.}$$

The acceleration of gravity in the latitude of London is 981 (cm.-sec. units), and 32·18 (ft.-sec. units).

The weight of 1 lb. produces in a mass of 1 lb. an acceleration of 32·18 feet per second per second in London. If we take as our unit force the force which produces unit (foot-second) acceleration in a pound mass, we see that it must be $\frac{1}{32{\cdot}18}$ of the weight of a pound in London. This unit is sometimes used by textbooks and examiners, and is called a **poundal**.

Roughly, 1 lb. weight = 32 poundals.

Suppose that we are using pound, foot, second units, we should find that, roughly

$$\text{Force in pounds weight} = \frac{1}{32} \times \text{mass in pounds} \times \text{acceleration in foot-second units.}$$

§ **48.** Newton's Second Law of Motion gives one way in which force is measured. We measure force by the effect it produces, the amount of change which it causes in the motion of a body in a given time. The amount of change itself depends on two things : (*a*) the mass acted upon, (*b*) the change produced in its speed. The time taken to produce the change is to be taken into consideration. For instance, how does the force of explosion of the charge of a rifle compare with the force of gravity? Suppose the bullet to be fired straight up into the air. Before the rifle is fired, the bullet is at rest, and afterwards it rises until the force of gravity has destroyed the motion imparted by the explosion. If we did not take the time into consideration, we might suppose the two forces to be equal, because one destroys the motion caused by the other. But that is not our way of defining *force.* The time must be considered. Suppose the barrel of the rifle to be 3 feet long, and the muzzle velocity, i.e. the speed with which the bullet emerges from the muzzle, to be 2,000 feet per second. The average speed inside the muzzle would then be 1,000 feet per second, and as the force of the explosive only acts on the bullet while it is inside the barrel, it only acts for $\frac{3}{1000}$ of a second, the time that the bullet takes to get outside the rifle. Afterwards this speed is gradually destroyed by gravity; leaving the resistance of the air out of account, it would be 62½ seconds before the bullet reached its highest point. Hence the weight of the bullet takes 62½ seconds to destroy the velocity created by the explosive in $\frac{3}{1000}$ of a second. Hence the average force of the explosive on the bullet is $\frac{1000}{3} \times 62\frac{1}{2}$ times the weight of the bullet, i.e. 20,833 times the weight of the bullet.

A rifle is not usually fired straight up into the air, so that the force of gravity does not as a rule oppose the force of the explosion directly. If the rifle is fired horizontally, the weight of the bullet pulls it out of the line in which it is travelling at first, and this amount of deflection increases with the time that the bullet takes to travel. We know that with increased muzzle velocity the rifle need not be pointed so high above the object aimed at; in other words, the elevation becomes less, and the bullet has a lower trajectory.

It is plain, then, that there are many things to be measured in any dynamical problem, i.e. any problem concerned with material objects in motion, and the forces acting on them.

We have to measure

(*a*) times,
(*b*) distances,
(*c*) masses,
(*d*) angles,
(*e*) velocities,
(*f*) accelerations, i.e. the rates at which velocities change,
(*g*) forces.

Many of these are not easy to measure. For instance, it is particularly difficult to measure small intervals of time with any accuracy, and this makes it hard to measure velocities and the rate at which they change. Again, in measuring forces, it is difficult to get rid of disturbing forces, such as friction of wheels, and resistance of the air, and this makes it impossible to verify the laws of force accurately without taking many elaborate precautions to get rid of these disturbing elements. If it were not so, it is probable that men would have discovered these laws before 1600 A.D.

The movements of the earth and the planets are not affected by friction or resistance to any appreciable degree. Hence Newton, formulating the facts discovered by Galileo, was able to work out a complete explanation of the facts discovered by Kepler and others on strict mathematical reasoning. Since his time the facts of astronomy have been given as sufficient proof of Newton's Laws; but Dynamics remains an experimental science, and those who are not good enough astronomers and mathematicians to understand the work of Kepler and Newton must base their conclusions on the work of Galileo or take Newton's Laws and verify them experimentally.

§ **49. Atwood's Machine.** Two masses, m_1 and m_2, are connected by a light string passing over a smooth pulley. What is the acceleration of each mass, and the tension of the string?

Let T = tension of string in dynes (Fig. 43).

If the mass m_1 is in grammes, its weight in dynes is $m_1 g$ where g in this latitude = 981.

The mass m_1 is acted on by its own weight downwards, and the tension of the string upwards, making $m_1 g - T$ downwards. If f is the acceleration in (cm. dynes -sec. units,) this gives

$$m_1 f = m_1 g - T.$$

Similarly for m_2, $m_2 f = T - m_2 g$.

$$\therefore\ (m_1 + m_2) f = (m_1 - m_2) g$$

$$f = \frac{m_1 - m_2}{m_1 + m_2} \cdot g.$$

Substituting for f, we get

$$T = m_2 (f + g) = m_2 \left(\frac{2m_1}{m_1 + m_2}\right) g$$

$$= \frac{2\, m_1\, m_2}{m_1 + m_2} \cdot g.$$

§ **50. Exercises.** (i) Masses of 24 and 21 grammes are connected as in Fig. 43. Find the acceleration, and the tension of the string (*a*) in dynes, (*b*) in grammes weight.

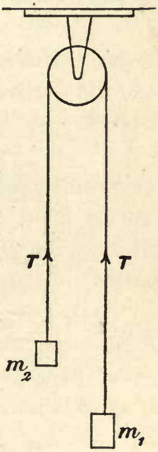


Fig. 43.

(ii) A mass *M* lb. is pulled on a smooth horizontal plane by a mass *m* lb. hanging over a pulley as in Fig. 36. Find the acceleration and the tension of the string in pounds weight.

If this arrangement is tried experimentally, as in § 41, allowance must be made for the force necessary to overcome friction. Again, if a roller or wheels are used as in Fig. 37, it will be necessary to make allowance for the energy of rotation of the wheels and pulleys (see § 106), as well as for the friction. For the sake of simplifying problems in examinations, the surfaces are frequently supposed to be really smooth, and the pulleys light, so that it is possible to neglect friction and moment of inertia.

(iii) A uniform chain of mass 28 lb. is attached to a mass of 1 cwt., and both rest on a smooth horizontal table. The other end of the chain is connected by a light string passing over a smooth pulley to a mass of 28 lb., which is allowed to descend. Find the acceleration, and the tension at the middle point of the chain. Fig. 44 shows the arrangement.

Let T be the tension of the chain at P in pounds weight. Then the force acting on the 28 lb. and the near half of the chain is $28-T$ lb. weight. The force acting on the remaining 126 lb. is T lb. Let $a=$ the acceleration in foot-second units.

The equations for the acceleration are :—

$k(14+28)\,a$ (ft.-sec. units) $= 28 - T$ pounds weight

$$\text{but} \quad k = \frac{1}{32{\cdot}18} = \frac{1}{g} \text{ (see § 47),}$$

$$\therefore \quad 42a = (28-T)g.$$

Also
$$126a = Tg,$$

$$\therefore \quad 168a = 28g.$$

$$a = \frac{g}{6} = \frac{32{\cdot}18}{6} = 5{\cdot}36 \text{ ft. per sec. per sec.}$$

$$T = \frac{126a}{g} = \frac{126}{6} = 21 \text{ lb. weight.}$$

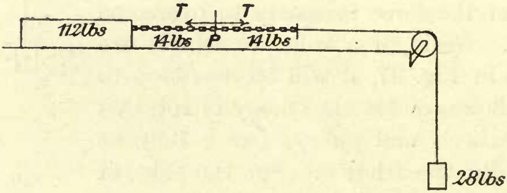


Fig. 44.

(iv) What force is required to change the velocity of a mass of 1,000 grammes from 12 cms. per second to 372 cms. per second in one minute? (*Lond. Matric.*)

(v) A cage weighing 4 cwt. is drawn up a shaft with constant velocity; state what is the tension of the rope, giving clear reasons for your answer. (*Lond. Matric.*)

(vi) A mass of 500 lb. moving in a straight line is observed to change its speed from 60 miles per hour to 50 miles per hour in 10 seconds. What force must have acted to produce the change?

(vii) A car weighing 10 tons carries 20 passengers, whose average weight is 150 lb. What is the total momentum when the speed of the car is 12 feet per second, and what resisting force must be applied to stop the car in 15 seconds?

(viii) What force, in pounds weight, will produce an acceleration of 8 feet per second per second in a mass of 12 lb.?

A man of weight 150 lb. stands on a platform; what pressure does he exert on it if the platform moves vertically with an acceleration of 2 feet per second per second (*a*) upwards, (*b*) downwards?

(ix) If the block of a steam hammer weighs half a hundredweight, and has moved through a distance of 8 inches under the action of a uniform force of 10,000 pounds weight, with what momentum is it then moving? (Name your units.)

(x) When a lift is at rest its floor is just strong enough to carry a load of 1 ton. If the lift is moving (*a*) with constant speed, (*b*) with an acceleration, can the floor support the same load?

(xi) If a force equal to 20 lb. weight acts upon a certain mass, it causes the mass to move with a velocity which increases at the rate of 30 feet per minute in every second. What acceleration would be caused by a force of 40 lb. weight acting upon the same mass?

(xii) In walking down the corridor of a train in the direction opposite to that of the train's motion, while the brakes are being applied on approaching a station, a sensation is experienced very analogous to that of walking up hill. Explain this. (*Lond. Matric.*)

(xiii) A body weighing W lb. is carried up in a lift which moves with an acceleration of 4 feet per second per second. State the direction of the force exerted upon it by the floor of the lift, and the weight of a body which this force could support at rest.

How would the result be modified if the lift ascended with uniform velocity?

(xiv) A body weighing W lb. is acted on by a force which could support a weight of W' lb. at rest. In what time will it acquire a velocity v feet per second?

A body is suspended by a cord from the roof of a railway carriage forming part of a starting train, and it is observed that the cord is inclined to the vertical at an angle whose tangent is $\frac{1}{8}$. In what time will the train acquire a velocity of 30 miles an hour?

(xv) A train of 300 tons moving with a speed of 40 miles an hour is brought to rest by friction in 2 minutes (steam being shut off). Find the magnitude of the frictional force, which is assumed to be constant.

(xvi) A mass P on a smooth horizontal table is connected with unequal masses Q and R by strings passing over smooth pulleys at opposite edges of the table. Find the acceleration.

If $Q = 7$ lb. and $R = 2$ lb., the system starting from rest describes 5 feet in the first second. If $Q = 4$ lb. and $R = 1$ lb. the system describes 4 ft. in the first second. From these data determine P and the acceleration of gravity.

(xvii) State Newton's Second Law of Motion, and apply it to the following problems :—

(*a*) A body weighing 80 grains, sliding down an incline of 30° from rest, describes 5 feet in the first second. Find the retarding force up the plane.

(*b*) A body of weight W is acted on by gravity and by a force $\frac{1}{2}\sqrt{3}\,W$ acting in a direction inclined at 30° to the upward drawn vertical. Find the direction in which the body begins to move.

(xviii) Two scale-pans, each weighing 2 ounces, are connected by a fine string passing over a smooth fixed pulley, and are hanging at rest. A weight of 15 oz. is placed in one scale-pan, and, simultaneously, a weight of 13 oz. is placed in the other scale-pan. Find the acceleration of each scale-pan, the tension of the string, and the pressures of the weights on the scale-pans.

(xix) Investigate the motion of two unequal weights connected by a cord which passes over a smooth fixed pulley.

A heavy string hangs at rest over a small smooth fixed peg. Show that if $\frac{198}{199}$ths of the string on one side be cut off, the pressure on the peg is instantaneously reduced to $\frac{1}{100}$th of its previous amount.

(xx) Two masses, one 960, the other 1,000 grammes, are connected by a cord, the mass of which may be neglected, and which passes over a smooth peg. Find how far the heavier descends in 10 seconds, and the tension in the cord.

[g may be taken as 980 in C. G. S. units.]

What difficulty would appear in attempting to determine the motion if the peg were replaced by a pulley turning without friction. (*Lond. Matric.*)

(xxi) A mass of 370 lb. is pulled up a rough slope of 1 in 20 (so that going 20 inches along the slope gives a rise of 1 inch) by a rope parallel to the plane, whose tension is 1,800 poundals (see § 47). If the resistance due to friction is 1,100 poundals, find the velocity acquired by the mass in 10 seconds. Give the velocity to the nearest foot per second.

CHAPTER V

MOMENTUM. NEWTON'S THIRD LAW. IMPULSE

§ **51. Momentum** is the 'quantity of motion' in a body, the mass of the body having to be considered as well as the velocity. A railway train and a pigeon may both be travelling 60 miles an hour; but the momentum of the former is by far the greater. If two masses, one double of the other, are moving with the same velocity, the momentum of the former is double that of the latter; again, if two equal masses have different velocities their momenta will be as their velocities. A mass of 1 ton moving at the rate of 1 foot per second has the same momentum as a mass of 1 lb. moving 2,240 feet per second. *We measure the momentum by multiplying the number of units of mass by the number of units of velocity.* On the C. G. S. system a kilogramme moving 1 metre per second has $1{,}000 \times 100$ units of momentum.

(i) What is the momentum (C. G. S. units) of 1 lb. moving at the rate of 1 foot per second?

§ **52. Mass and Momentum.** Newton experimented with balls hung by strings as in Fig 45, allowing them to fall from equal heights against one another, and arrived at certain conclusions. The material of which the balls are made makes a great deal of difference to the results. Some will rebound almost to the same height, e.g. balls made of glass, ivory, or steel. If the balls are made of a substance like putty, there will be hardly any rebound. In arrangements such as Hicks's Ballistic Pendulum (see Hicks's *Dynamics*, p. 24), the two masses can be made to stick together after impact by various devices. This apparatus, described in Hicks's *Elementary Dynamics*, is supplied by most makers of physical appliances ; and it is not difficult to make. One plan for making the carriers stick together is to make them of soft wood, carrying sharp projecting points which will penetrate the wood and stay there. The wooden carriers can be given any required velocity by pulling them back different distances, the velocity depending only on the height from which they fall, and they should be set so as to come into contact at the lowest point of their movement. It will be found that equal masses coming together with equal velocities destroy each other's motion. This should be tried with different loads and different velocities.

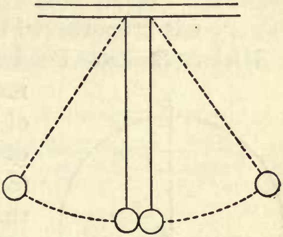


FIG. 45.

It has been pointed out that the weight of a body varies with its position on the earth's surface (§ 6). However, we know that equal masses have equal weights at the same place, and we 'weigh' masses, i.e. we compare them by comparing their weights. The ballistic pendulum gives another way of comparing masses. Masses are equal if they destroy each other's motion when they come together with equal and opposite velocities and stick together.

Compare various equal masses in this way.

It should be noticed that the velocity of the carriers at the lowest point is closely proportional to the horizontal distance to which that velocity will carry them when there is no impact.

To show this let O be the point of support of the pendulum, B the point from which it is allowed to fall. Then if v be the velocity at A, the lowest point, $v^2 = 2gh = 2g \times AC$ (Fig. 46).

But $AC \, . \, CD = BC^2$

$$\therefore \quad AC = \frac{BC^2}{CD} = \frac{BC^2}{2R} \text{ nearly.}$$

$$\therefore \quad v^2 = \frac{g}{R} \times BC^2 \text{ when } R \text{ is large in comparison with } BC.$$

$\therefore$ v is proportional to BC for small displacements.

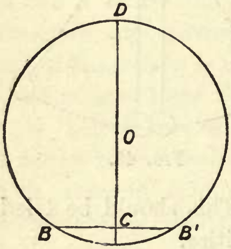


Fig. 46.

Hicks's Ballistic Pendulum can now be used with unequal masses. Take, for example, masses of 50 and 100 grammes. The weight of the carriers must, of course, be found and reckoned in. In order that the masses may be at rest after collision, the 50-gramme mass must be given just double the velocity of the other.

The vibrating spring already described, § 42, may be utilized in examining the momenta of two trolleys before and after impact. The trolleys can be loaded with any masses, and can be sent to meet each other with any velocities. If well made they run perfectly true, and can be set to run into each other without rails. One trolley should carry projecting needle points so that it may not rebound from the other trolley after impact. The arrangement is shown in Fig. 47. The lengths of the waves immediately before and after impact supply the necessary information about the velocities.

§ 53. Conservation of Momentum. If two unequal masses are to bring each other to rest, their **momenta** must be equal and opposite.

If direction is expressed by signs (+ and −), we may say that the total momentum of two such masses coming together is zero both before and after impact. This leads to the next step, which is to show that the total momentum of two bodies which collide is unchanged by the impact.

Let one of the trolleys (or carriers) with any 'weight' be at rest, and allow the other one to impinge on it from a marked distance. The two trolleys after impact will move on in the same direction. If the pendulum is being used, note the point to which the carriers swing. This gives a measure of the velocity of the whole system after impact. It will be found that the momentum immediately after impact, measured by multiplying the combined masses by the velocity, is equal to the momentum of the first mass immediately before impact.

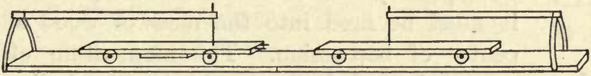


Fig. 47.

Further experiments should be made with different masses and different velocities, taking care to ascribe the right signs to the velocities.

This law of the **Conservation of Momentum** may be expressed in the form $m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$, where m_1, m_2 are the two masses and u_1, u_2 their velocities before, v_1, v_2 their velocities after impact, due regard being paid to signs.

Stated in words, this is :—

The momentum, in a given direction, of two masses

which collide is the same after impact as before (proper regard being paid to signs).

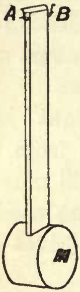


FIG. 48.

Robins's Ballistic Pendulum (Fig. 48), invented by Benjamin Robins, 1742, consists of a mass of wood *M*, which is suspended from a hinge *AB*. A point can be found in the wood at which a blow can be given which will not produce a jar on the hinge, just as there is a point in the blade of a cricket bat or the face of a golf club where the impact of the ball causes no jar on the hand. This point, called the **centre of percussion**, is at a distance from *AB* equal to the length of the equivalent simple pendulum (see note, p. 239). To measure the velocity of a bullet, it must be fired into the mass of wood at the centre of percussion. The momentum of the whole mass, consisting of the wood with the bullet embedded in it, can then be calculated from the distance to which the pendulum swings. The pendulum carries a tape which passes through a light clip, so that this distance can be easily measured.

(i) A bullet weighing $\frac{1}{2}$ oz. is fired into a ballistic pendulum weighing 56 lb., and moves it through a distance which corresponds to an initial velocity of 1 foot per second. What was the velocity of the bullet?

The law of the Conservation of Momentum is strictly in accordance with Newton's First Law, the law of **Inertia** (§ 11). For if we consider the two masses which collide as forming one mass in which the relative positions are changing, we see that there is no change in the movement of the mass as a whole.

A man on a perfectly smooth horizontal surface can get no outside force to move him along the surface, as there is no friction. He may raise himself up and down, slide his feet about, double himself up, and wave his arms, but his centre of gravity (see § 85) will always be vertically above the same point of the surface. Of course, a truly smooth surface is unattainable, and it may be left to the student as a breakfast-table problem to devise a method for the man to get off the imaginary table.

§ **54. Newton's Third Law.** In considering the movements of the two parts of a compound mass, we are led, by experiments with ballistic pendulums such as those described, to the conclusion stated in Newton's Third Law of Motion : **'To every action there is an equal and opposite reaction.'** This is familiar enough to be regarded as obvious in such instances as a book resting on a table. The book presses on the table, and the table presses on the book with an equal and opposite pressure. In the tension of a rope, the stress at any point P in the rope (Fig. 49) is pulling the right-hand portion of the rope towards the left, and an equal and opposite stress is pulling the left-hand portion of the rope towards the right.

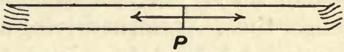


FIG. 49.

In the case of bodies in motion the principle is not so obvious. It will be necessary to explain the method of measuring forces which act for a very short time, too short to be measured. Such forces as the blow of a cricket bat on a ball do not differ from forces like gravity except in magnitude and in not being uniform.

'This conception of Gravity as a Uniform Force—constantly and equally *increasing* the velocity of a descending body—will become clear by a little attention, but it undoubtedly presents difficulty at first' (Whewell). Descartes could not grasp this: 'It is certain,' he said, 'that a stone is not equally disposed to receive an increase of velocity when it is already moving very quickly, and when it is moving slowly.' But he did not realize that the movement of the stone does not take it away from the force of gravity, as it does from the pressure of the hand when we throw the stone. The hand cannot follow the stone in its flight, and so the velocity of the stone does not increase after it has left the hand. It is, undoubtedly, an advantage in throwing for the pressure to be continued as long as possible. What makes a spliced cricket bat drive? The elasticity of the handle allows the ball to be in contact with the bat for a longer time. Again, in driving a golf ball the 'follow through' of the club-head is a most important item, because the contact between the ball and the club is maintained for some time, even while the ball is already travelling with considerable speed.

§ **55. Impulses.** When the force acts for a very short time, as in the case of a ball struck with a bat or a club, it is difficult to measure the *rate* at which the velocity is altered during the short time that the impact lasts. The *total change* of velocity can be measured, and if we knew the time the impact lasted we could divide the total change by this and so obtain the average acceleration, which, multiplied by the mass moved, would give the average force. It is usual, however, to multiply the total change of velocity by the mass, and call this the **impulse**.

But if we do not take the time into consideration, we may be led into very erroneous notions as to the nature of an impulse. When a cricket ball is caught, its momentum has to be destroyed by the pressure of the hands. If the hand is so placed as to be able to ‘give,’ and allow time for the destruction of the momentum, the pressure of the ball on the hand during this time is not painful. The yielding of the fingers helps towards this, so that astonishing catches are made without discomfort if the ball happens to come in the right place. But if it strikes the top of a partially closed finger, which cannot yield much, the result is frequently a broken joint. Similarly a very little paper between plates packed together may just give enough yielding space to prevent the crockery from being broken by a sudden jar. *To make the force at any instant smaller by prolonging the time occupied by the impulse* is the way to prevent damage, and this principle is applied in the use of sandbags and earthworks as defences. Examine the face of a wooden golf club that has been used with rubber-filled balls and compare it with one that has been used with the plain gutty, and the advantage of a prolonged impact between the club-head and the ball will become obvious.

§ **56. Instances of Action and Reaction.** A bullet is fired from a rifle. The rifle recoils. The explosion of the powder produces an expansive force inside the barrel which pushes the bullet forward and pushes the gun back equally. Not considering the time during which the bullet is inside the barrel, we simply measure the *impulse* of the explosion which is the momentum of the bullet and also the momentum of the rifle. If the mass of the rifle be 1,000 times that of the bullet, and the latter have a velocity of 1,000 feet per second, the rifle recoils with a velocity of 1 foot per second.

The momentum of the bullet and rifle, regarded as a single *system*, remains unaltered by the firing of the rifle; for the movements take place in opposite directions, and the momenta have therefore opposite signs, their sum being zero. The centre of gravity of the two bodies, regarded as one, remains vertically over the same spot for the short time during which the movements of bullet and rifle go on unchecked. Does this agree with Newton's First Law?

Connect two weights *A* and *B* by a spring-balance. Then connect another spring-balance to *A* and let it support both weights (Fig. 50). Is the difference between the readings of S_2 and S_1 equal to the weight of *A*? This experiment shows clearly that the spring-balance S_1 is pulling *A* down, though it is pulling *B* up; and allowing for the weight of S_1 itself, the downward pull on *A* is equal to the upward pull on *B*. This is another example of the law illustrated by the recoil of the gun, that, whenever there is a force between two bodies, it acts equally both ways. As Newton puts it, 'Action and reaction are equal and opposite.'

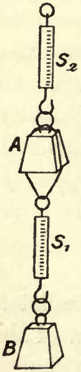


Fig. 50.

§ **57. Exercises.** (i) A horse is drawing a cart. What is the force pulling the cart forward? Is the same force pulling the horse back? Why is the horse not pulled back? Considering the horse and cart as one moving body, what force sends them forward? If the horse and the cart each weigh a ton, how does this force compare with the force the horse requires to move himself alone without the cart at the same pace? How does it compare with the pull or tension of the traces connecting the horse to the cart?

(ii) A rifle bullet weighing 25 grammes strikes horizontally a block of wood whose mass is 15,000 grammes lying on smooth ice, with velocity 500 metres per second, and remains embedded in the block. Find the velocity with which the block moves.

(iii) A shot weighing 10 lb. is discharged from a gun weighing 10 tons. The shot escapes from the gun, with a velocity of 2,000 feet per second, $\frac{4}{5}$ of a second after the powder is fired. Calculate the velocity of the gun, at the instant the shot leaves it, assuming that its motion is unimpeded.

What average force would it be necessary to exert upon the gun to keep it from changing its position during discharge.

(iv) Enunciate Newton's Third Law of Motion, and give two examples of it. A rope hangs from a spring-balance which is attached to a fixed beam. A man of given weight hangs on the rope half-way between the floor and the beam. Represent in a diagram the forces acting upon the man (*a*) when he is at rest, (*b*) when he is climbing the rope. How would the indication of the balance be affected in the latter case?

(v) Define force and momentum. How is the force acting upon a body related to the change of momentum produced by it?

A jet discharges 100 lb. of water per minute with a velocity of 50 feet per second perpendicularly against a wall. How much momentum is destroyed every minute by the wall, and what force does the water exert upon the wall?

(vi) A deep pot with tissue paper, on which rests a weight, stretched over the top is placed on one pan of an ordinary balance and counterpoised. The paper suddenly gives way and the weight falls to the bottom of the pot. Carefully explain the behaviour of the balance after the breaking of the paper.

(vii) If a shot of 140 lb. is fired from a 5-ton gun with a muzzle velocity of 1,600 feet per second, and the recoil of the gun is taken up by causing it to ascend an inclined plane, through what *vertical* height will the gun rise? (Neglect the weight of the powder charge.)

(viii) A light rope passes over a smooth pulley, and from one end of the rope a monkey hangs, from the other end a weight equal to that of the monkey. The monkey begins to climb the rope. What happens ? How is the case affected by friction in the pulley ?

CHAPTER VI

PARALLELOGRAM AND TRIANGLE OF FORCES

§ **58. Representation of a Force.** In making a diagram showing a force it is necessary to show the following :—

(1) The point at which the force acts.

(2) The line along which it acts.

(3) The sense in which it acts along this line ; i.e. whether it is pushing or pulling.

(4) The size of the force.

All these can be shown by drawing a straight line with an arrow.

(i) Draw on your paper two straight lines crossing at right angles to represent the directions north, south, east, and west. Then, taking a distance of 1 inch (or 1 cm.) as representing the weight of 1 lb., show on your diagram the following forces acting at the point of intersection of your two straight lines :—

A pull of 4 lb. weight in a NW. direction.
A pull of 2½ lb. weight in a SE. direction.
A push of 5 lb. weight in a SW. direction.
A push of 3 lb. weight in a NE. direction.

(ii) The words **tension** and **thrust** (or **pressure**) are used for pull and push. A bar meant to resist tension is called a **tie**, and one meant to resist thrust or pressure is called a strut. What are the bars in Fig. 51, and what are the forces acting on them? (Scale 10 lb. weight to the inch.)

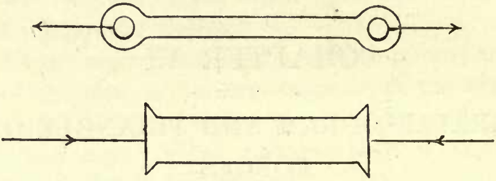


FIG. 51.

§ **59. Forces in equilibrium.** When a body is at rest, or moving uniformly in a straight line, there must be either no force acting on it, or else the forces acting on it must balance each other. This latter alternative is the only one with which we are concerned, as it is impossible to arrange that a body should be acted on by no force at all. When the forces acting on a body balance each other, they are said to be **in equilibrium.**

§ **60. Simple case of two forces in equilibrium.** Take a perfectly round ball of steel or glass and place it on a smooth horizontal table. A plate of glass adjusted on three levelling screws (see Appendix; Pye's Little Jack) would be suitable for this. We have then two forces in equilibrium, and it is obvious that the two forces acting on the ball are :—

(*a*) Its weight.

(*b*) The upward pressure of the table.

(i) Draw a figure showing the directions of these forces, and the points at which they act. How can you represent in your figure the fact that the two forces are equal?

(ii) Give the ball a horizontal movement. The two forces are still in equilibrium, and do not interfere with the motion of the ball. Does the ball stop, or turn? If so, what is the cause of this?

(iii) Another simple case of two forces in equilibrium occurs in the case of a ball hung up by a string. The two forces are the weight of the ball, and the tension of the string. There is one position of equilibrium, and one only. Make a drawing of this position. Fig. 53 shows another position.

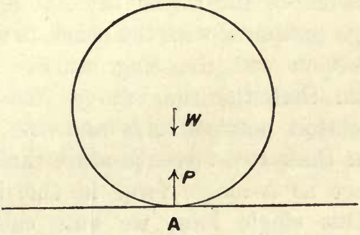


Fig. 52.

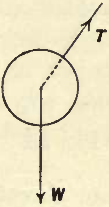


Fig. 53.

(iv) The case of an object like a book, or a block of wood resting on a table, will be considered later, as the upward pressure of the table is applied at several different points, wherever the block is in contact with the table; so that we have the weight of the block balanced by several upward forces. Write down the conditions that two single forces should be in equilibrium, and illustrate by a diagram.

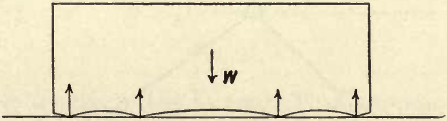


Fig. 54.

§ 61. When are three forces in equilibrium? In Fig. 55 a small ring *A* is shown with two strings pulling in the directions *AB*, *AC*. In what direction will the ring move, supposing the two forces to be equal? Suppose that one of the forces, say *AC*, is very large compared with the other, in which direction will the ring move? It is plain that the ring moves in some direction between *AB* and *AC*, and that these two forces produce the same effect as a single force in this intermediate direction. This single force we may call the **resultant** of the two forces, and, to find its direction and magnitude by experiment, we must find the direction and magnitude of the force which will keep the other two in equilibrium, and may be called the **equilibrant**.

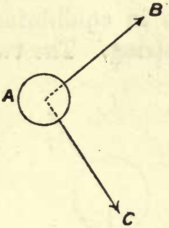


FIG. 55.

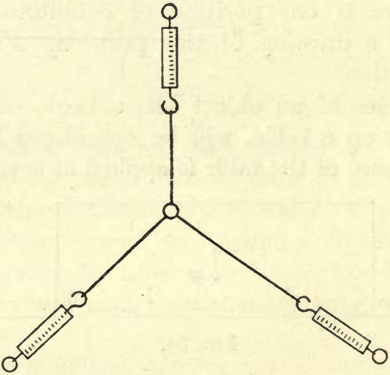


FIG. 56.

This may be done in various ways. One way is to attach three strings to the ring and connect them to spring-balances fastened to pegs or nails at different corners of the table. If the string leading from one of the spring-balances to its peg is pulled tight and hitched on to the peg, the ring will be kept at rest by the action of three forces. Put a sheet of paper underneath the ring, and mark on it the centre of the ring, and the directions of the three strings, writing against each the number of pounds registered by the spring-balance attached to the string. Then remove the paper, and rule lines from the centre of the ring in the directions marked, making these lines proportional in length to the forces. Thus suppose the directions of the strings to be AB, AC, AD, and the spring-balances attached to these strings to show 9 lb., $12\frac{1}{2}$ lb., 10 lb. respectively. We would make (see Fig. 57)

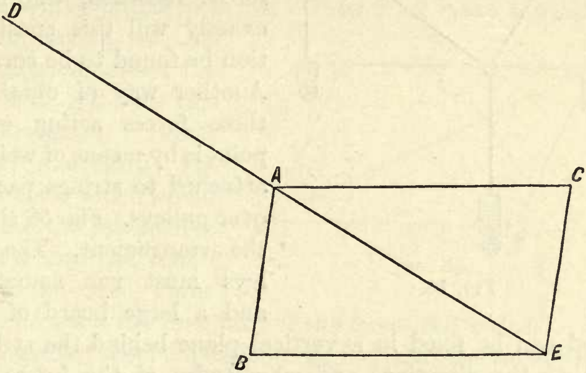


Fig. 57.

$$AB,\ AC,\ AD \text{ in the proportion } 9 : 12\tfrac{1}{2} : 10, \text{ or } 18 : 25 : 20,$$

$$\text{e.g. } AB = 1{\cdot}8 \text{ in.},\ BC = 2{\cdot}5 \text{ in.},\ AD = 2 \text{ in.}$$

Then these lines represent the three forces acting on the ring at A. If by means of set squares we complete the parallelogram *ABEC*, by drawing through *B* a parallel to *AC* and through *C* a parallel to *AB*, then we shall find that the diagonal of this parallelogram, viz. *AE*, is equal and opposite to *AD*, i.e. *AE* and *AD* are in the same straight line, and $AE = AD$. The same thing will be found to be true for the parallelogram formed by *AB* and *AD*; its diagonal will be equal and opposite to *AC*; and the diagonal of the parallelogram formed by *AC* and *AD* will be equal and opposite to *AB*. The more accurately the directions are marked, and the magnitudes of the forces recorded, the more exactly will this construction be found to be correct. Another way of obtaining three forces acting on a point is by means of weights attached to strings passing over pulleys. Fig. 58 shows the arrangement. The pulleys must run smoothly, and a large board of soft wood can be fixed in a vertical plane behind the strings, so that the directions and magnitudes of the forces can be marked on a sheet of paper pinned to the board. Pulleys for this purpose can be obtained of any maker of physical apparatus. They are best made of aluminium, with clamps for fixing them to any point on the edge of the board : or simple wooden pulleys two or three inches in diameter, with bushes in the middle large enough to allow them to run freely on French nails or bradawls, can be easily made. The bradawl can then be inserted at any required point of the board.

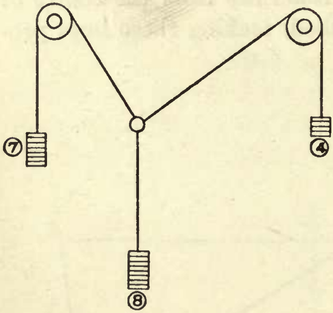


FIG. 58.

The experiment should be repeated with several different sets of forces, so that the student may satisfy himself of the truth of the following statement:—*If three forces acting at a point are in equilibrium, and if these forces are represented in magnitude and direction by straight lines, each of these straight lines is equal and opposite to the diagonal of the parallelogram formed by the other two.*

§ **62. Parallelogram of Forces.** From this follows the important law known as the parallelogram of forces.

Suppose *AB* and *AC* (Fig. 59) to represent *completely* two forces acting on the point *A*. Then if we make a parallelogram, having *AB*, *AC* for adjacent sides, the diagonal *AE* of this parallelogram represents *completely* the single force which would produce the same effect as *AB* and *AC* combined, in other words *AE* represents the **resultant** of the forces *AB* and *AC*. The experimental proof depends on the fact that the force which keeps *AB*, *AC* in equilibrium is *equal and opposite to AE.*

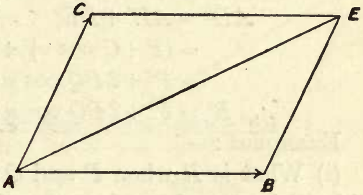


FIG. 59.

Thus we find that the ring in Fig. 55 would move along the diagonal of the parallelogram formed by *AB*, *AC* as if there was a single force proportional to this diagonal pulling it in this direction.

The parallelogram of forces is merely one case of the more general **Parallelogram of Vectors** (see § 15), for a force is a **vector**, i.e. a quantity which has direction as well as magnitude, and can therefore be represented completely by a straight line.

(i) Suppose that in Fig. 59 the sense (§ 58) of one of the forces, say *AB*, was changed, how would the figure be altered; how would it be altered if the sense of both forces was changed? Draw diagrams.

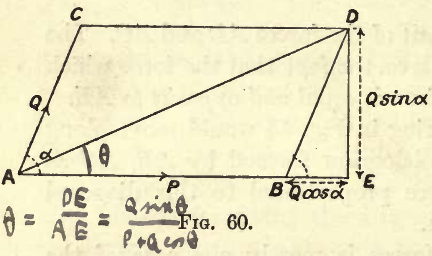
§ **63.** **Formula.** Let *P* and *Q* be two forces acting at *A* and making an angle *a*.

To find *R*, their resultant:—

$$
\begin{aligned}
AE &= P+Q\cos a \text{ (Fig 60)}\\
AE &= P-Q\cos(180^\circ-a) \text{ (Fig. 61)}\\
&= P+Q\cos a\\
DE &= Q\sin a \text{ (Fig. 60)}\\
&= Q\sin(180^\circ-a) = Q\sin a \text{ (Fig. 61)}\\
AD^2 &= AE^2+DE^2\\
&= (P+Q\cos a)^2+(Q\sin a)^2\\
&= P^2+2PQ\cos a+Q^2(\cos^2 a+\sin^2 a)\\
\therefore\quad R^2 &= P^2+2PQ\cos a+Q^2.
\end{aligned}
$$

Examples:—

(i) What is *R* when *P* and *Q* are at right angles? Find this (*a*) by the above formula, (*b*) by a separate figure.



$\tan\theta = \frac{DE}{AE} = \frac{Q\sin\theta}{P+Q\cos\theta}$

FIG. 60.

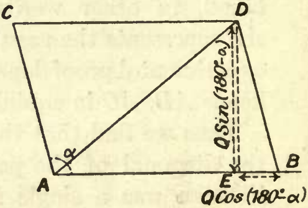


FIG. 61.

(ii) Find *R* when $a = 0^\circ$. Illustrate by a figure.

(iii) Find *R* when $a = 180^\circ$. Illustrate by a figure.

(iv) Forces of 10 lb. and 7 lb. act on the same point in directions north and east respectively. Find the magnitude and direction of their resultant, both by a figure drawn to scale, and by calculation.

(v) Find by drawing the magnitude and direction of R in the following cases :—

(1) $P = 10$, $Q = 12$, $a = 63°$.
(2) $P = 70$, $Q = 60$, $a = 140°$.
(3) $P = 7{\cdot}5$, $Q = 11{\cdot}2$, $a = 105°$.

Verify by calculation.

§ **64. Resolution of Forces.** It is plain that, if two forces represented by the sides of a parallelogram can be replaced by the single force represented by the diagonal of the parallelogram, then any single force may be replaced by two others, if only we make a parallelogram of which the single force is the diagonal. Thus in Fig. 62, let AB represent any force. Through A and B draw any parallels AC, BD, and also any other parallels AD, BC. Then $ACBD$ is a parallelogram, and AB can be replaced by the two forces AD, AC

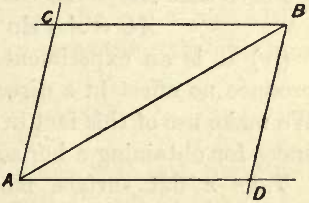


Fig. 62.

In practice the parallelogram is always right-angled. To take a practical example: Fig. 63 shows a truck on a line of rails. A horse is harnessed to the truck, and pulls it from *outside* the rails. It is obvious that the full force of the horse's pull AB does not go towards moving the truck

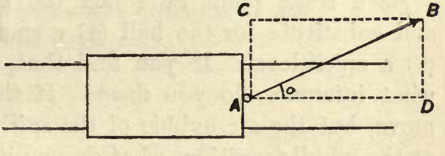


Fig. 63.

along the rails. Some is being wasted in sideways pressure against the rails.

(i) In what direction should the horse pull so that none of his force is wasted?

(ii) In what direction should he pull so that all of his force is wasted?

(iii) If AB makes an angle a with the direction of the rails, we can make a right-angled parallelogram, of which AB is the diagonal, and having its sides along the rails, and at right angles to them. Then AB can be replaced by the forces AD, AC.

Prove that $AD = AB \cos a$.

$$AC = AB \sin a.$$

(iv) It is an experimental fact that a single force can produce no effect in a direction at right angles to its own. We make use of this fact in the spirit-level and other appliances for obtaining a horizontal surface.

Take a flat surface mounted on levelling screws. A clean glass plate mounted on three screws as in § 60 will suit for this experiment. Now put a steel motor-car bearing on the plate, and adjust the levelling screws until the ball will rest in any position. If you find that it will remain still at some points on the plate and not at others, what do you infer?

Mark some point at which the ball remains at rest, and substitute for the ball (a) a small drop of mercury; (b) a spirit-level. If you find that the mercury moves, what inference do you draw? If the mercury does not move, but the air bubble of the spirit-level is not exactly at the middle position, what do you infer from this?

§ **65. Inclined Plane.** A smooth flat surface which can be levelled and then tilted to any required angle is required, also a heavy, smoothly running roller, or a small truck. Fig. 64 shows a possible arrangement. A spring-balance, or a weight hung over a pulley, will serve to measure the force up the plane. If a scale-pan is used, its weight must be known and allowed for. The surface of the plane should be perfectly horizontal before the tilting block is inserted. The angle of tilt can be measured with a protractor, or its sine found by measuring the distances AB and AC, and finding the ratio $\frac{AB}{AC}$.

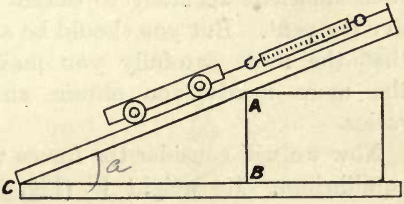


FIG. 64.

Find the weight of the roller W.

Find what force P_1 up the plane will just prevent it from rolling down, and what force P_2 will just not move it up. These are probably not quite the same, owing to friction of the roller; but the average of the two $\frac{1}{2}(P_1+P_2)$ may be taken as the force required to support the roller on the plane.

Let this force be P. Find the ratio $\frac{P}{W}$.

Repeat these observations at different angles.

Tabulate your results thus :—

Weight of cylinder = 250 g.					
a = angle of plane.	Height AB.	Length AC.	Sin $a = \frac{AB}{AC}$.	P	$\frac{P}{W}$.
32°	7·5 cm.	13·4	0·56	140 g.	0·56

You will probably find a difficulty in measuring $\frac{P}{W}$ and sin a with sufficient accuracy to obtain results agreeing within five per cent. But you should be able to convince yourself that the more carefully you make your measurements, the more nearly you obtain an equality in the two ratios.

Now we will consider the forces which keep the roller in equilibrium. Its weight W (Fig. 65) acts vertically, and can be represented by the straight line AB. Make the rectangle $ACBD$ with AC along the plane, and AD at right angles to it. Then the weight can be resolved into two components, AC and AD, and of these AC is the force pulling the roller down the plane, and AD the force pressing it against the plane. From the figure you should be able to prove to your own satisfaction that

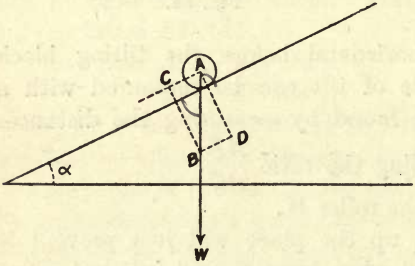


FIG. 65.

$$BAD = a;$$

$$\frac{AC}{AB} = \sin a; \therefore AC = W \sin a.$$

Thus to keep the roller at rest on the plane, we must apply a force P up the plane equal and opposite to AC. In other words $\frac{P}{W} = \sin a$.

So far we have not considered the other component of the weight, which is equal and opposite to the pressure of the plane on the roller. Let us call this pressure R. Then the three forces acting on the roller, which keep it at rest, are P, R, W (Fig. 66).

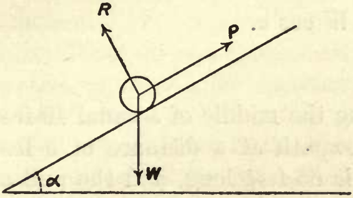


FIG. 66.

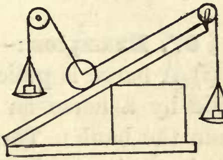


FIG. 67.

Copy the figure, but show the two components of W as in Fig. 65, and show that the ratio $\frac{R}{W} = \cos a$. Verify this experimentally by finding, either with a spring-balance or a pulley and weight as shown in Fig. 67, what force applied in a direction at right angles to the plane will just lift the roller off the plane. As before, the weight of the scale-pan must be found, and allowed for.

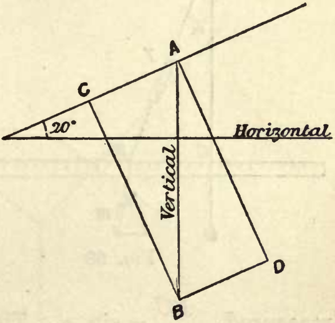


FIG. 68.

§ 66. Graphical Solutions. A truck weighing 5 tons is supported on an incline of 20° by a rope pulling up the incline. Find the tension of the rope and the pressure on the incline.

Fig. 68 indicates the construction. Let 1 inch represent 1 ton. (The scale had better be large, provided the construction is correctly made.) Take AB vertical and 5 in. long, and draw BD,

BC parallel and perpendicular to the incline. Measure AC, AD, and verify your results from the formulæ

$$P = W \sin \alpha.$$
$$R = W \cos \alpha.$$

§ 67. Examples :—

(i) A barge is pulled along the middle of a canal 40 feet wide by a horse on the towpath at a distance of 5 feet from the bank. The rope is 65 feet long, and the pull of the horse is equal to 1 cwt. Find by calculation and by construction the force moving the barge along, and the force pulling it towards the bank. How is the latter counteracted?

(ii) Find the force P necessary to pull a weight W suspended from A (Fig. 69) through an angle α from the vertical. Find also the tension of the string AB. This should be verified by experiment for several different positions. P and T can be measured by spring-balances as shown in the figure, and the distances BC, BA can be measured. $\frac{BC}{BA} = \sin \alpha$. The vertical position AC must be carefully marked. Tabulate your results as in § 65.

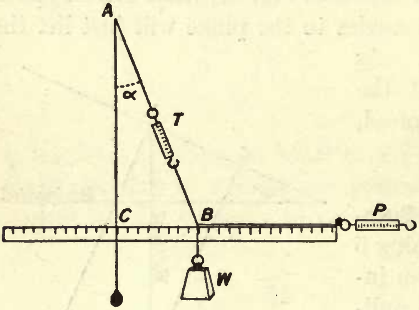


FIG. 69

(iii) A string with one end fixed runs horizontally to a pulley, passes round the pulley, and carries a weight of 56 lb. at the other end. Find the thrust of the pulley on its axle, neglecting the weight of the pulley and the cord. Give its magnitude to two significant figures, and state its direction.

(iv) What do you understand by the expression *the resultant* of two forces?

Find the resultant of the two forces shown in Fig. 70, acting at the point *P*.

(v) A body is acted on by two equal forces at right angles, one pushing, the other pulling. If the body is at rest at first, in what direction will it begin to move?

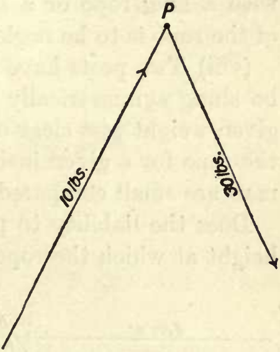


FIG. 70.

Would a body of twice the weight begin to move more quickly or less quickly?

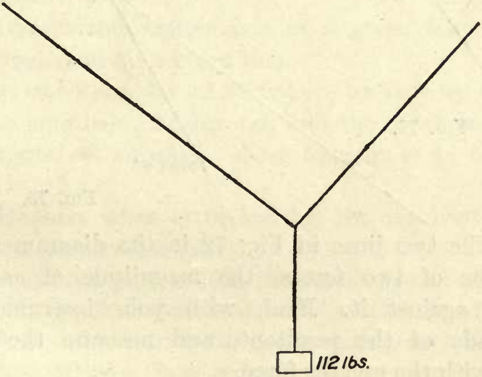


FIG. 71.

(vi) A weight of 112 lb. is supported by two strings, as in Fig. 71. Find the pull in each string.

(vii) In towing a boat along a river, is it easier to tow with a long rope or a short one, and why? (The weight of the rope is to be neglected.)

(viii) Two posts have been put up, and a hammock is to be slung symmetrically between them to carry a man of given weight just clear of the ground. Find the tension of the rope for a given inclination α. (The dimensions of the man are small compared to the length of the rope.)

Does the liability to pull down the posts depend on the height at which the rope is fixed to the posts?

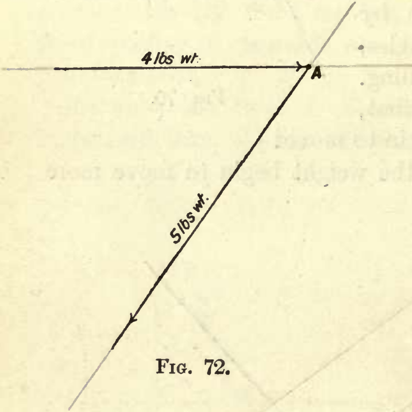


FIG. 72.

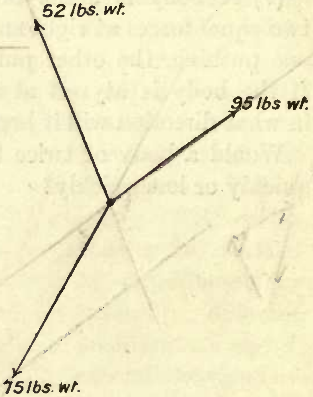


FIG. 73.

(ix) The two lines in Fig. 72 in the diagram show the directions of two forces, the magnitude of each being written against it. Find, with your instruments, the magnitude of the resultant, and measure the angle it makes with the greater force.

(x) Find the resultant of the forces shown in Fig. 73. Write down the magnitude of the resultant and the angle it makes with the force of 52 lb. weight.

(xi) Three forces P, Q and R, whose values are 10, 20, and 30 lb. weight respectively, act at a point in the directions shown in Fig. 74. Find the resultant, and measure the angle which it makes with the force Q.

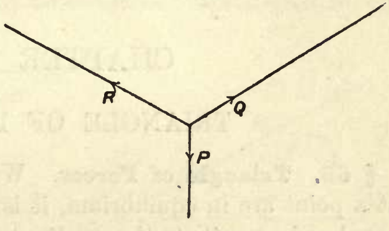


FIG. 74.

(xii) Enunciate the *parallelogram of forces*, and carefully describe the details of an experiment for demonstrating its truth.

OA, OB, OC are three equal straight lines representing three forces acting at the point O. The angle AOB being 90° and BOC being 45°, both measured in the same direction of rotation, determine the magnitude of the resultant of the three forces, and indicate its direction graphically. (*Lond. Matric.*)

(xiii) Obtain the components of a given force parallel and perpendicular to a given line.

A body which weighs 12 lb. is kept at rest by means of two cords, one being horizontal, and the other inclined to the horizontal at an angle whose tangent is $\frac{3}{4}$; find their tensions.

(xiv) Explain what is meant by the resolved part of a force in a given direction.

If the resolved part is half the force, what is the angle between the given direction and that of the force?

(xv) Find what horizontal force acting on the bob of a pendulum 80 cms. long will draw it 40 cms. out of the vertical. The weight of the rod of the pendulum may be neglected, the bob weighs 500 grms. (*Lond. Matric.*)

CHAPTER VII

TRIANGLE OF FORCES

§ **68. Triangle of Forces.** When three forces acting at a point are in equilibrium, it is plain that each force is equal and opposite to the resultant of the other two. When we represent them by straight lines as in Fig. 75, we see that each force is equal and opposite to the diagonal of the parallelogram formed by the other two. Also, if we take any one of the six triangles shown in Fig. 75, its sides are proportional to the three forces P, Q, R; and not only so, but the directions of the three forces follow each other round the triangle *in order*, either clockwise, as in Fig. 76, or counter-clockwise, as in Fig. 77. From this we obtain what is known as the **Triangle of Forces**, which may be stated thus:—

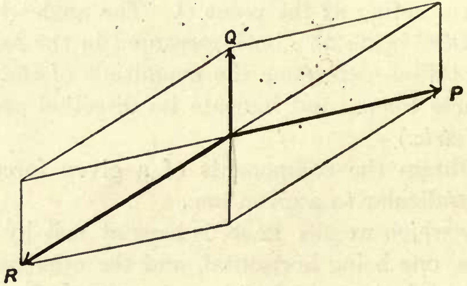


Fig. 75.

If three forces acting at a point are in equilibrium,

they may be represented (in magnitude and direction only) by the sides of a triangle taken in order.

Of course, they cannot be *completely* represented by the sides of a triangle, as these do not all pass through the same point.

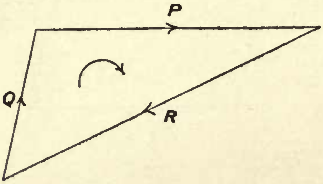


FIG. 76.

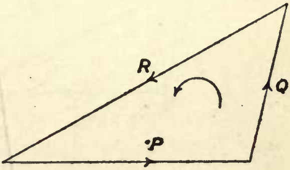


FIG. 77.

(i) The converse of this proposition is also true. State the converse.

§ **69. Lami's Theorem.** In Fig. 78, P, Q, R are supposed to be in equilibrium, and to be proportional to the sides of the triangle ABC.

Now these sides are proportional to the sines of the opposite angles.

Hence
$$P : Q : R = a : b : c$$
$$= \sin A : \sin B : \sin C.$$

But it is plain from the figure that since the sides of the triangle are parallel to the directions of the forces, the angles between these directions are as marked in the figure.

Also
$$\sin (180^\circ - A) = \sin A$$
$$\sin (180^\circ - B) = \sin B$$
$$\sin (180^\circ - C) = \sin C$$
$$\therefore \quad P : Q : R = \sin \widehat{QR} : \sin \widehat{RP} : \sin \widehat{PQ}$$
(where $\widehat{QR}$ means the angle between the directions of the forces Q and R, &c.).

Hence when three forces acting at a point are in equilibrium, each is proportional to the sine of the angle between the other two.

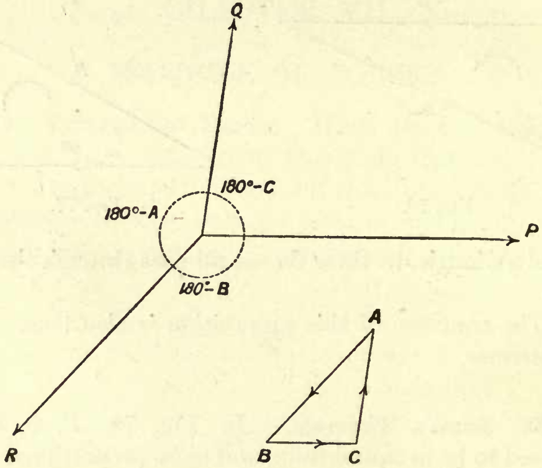


Fig. 78.

(i) The bob of a pendulum weighing 10 lb. is pulled aside by a horizontal string until its thread makes an angle of 20° with the vertical. Find the pull of the horizontal string, and the tension of the pendulum thread, both by construction and by calculation.

(ii) Show that if the angle be 30°, the tension of the pendulum thread must be twice the horizontal force.

§ 70. Polygon of Forces. Triangle and Polygon of Vectors. It is plain from Fig. 79 that if we take the two forces P and Q and represent them (in magnitude and direction only) by the sides AB, BC of the triangle ABC, then AC (not CA) represents their resultant. The same is true of all vectors (see § 15).

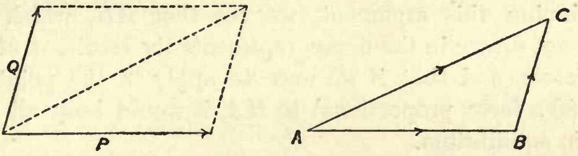


FIG. 79.

We may thus say that the resultant of two forces (or the sum of two vectors) can be obtained by drawing the two vector lines *consecutively* (as AB, BC), and then completing the triangle, and taking the third side, but *not* consecutively (as AC).

The force (or vector) represented by CA may be called the equilibrant (as opposed to resultant).

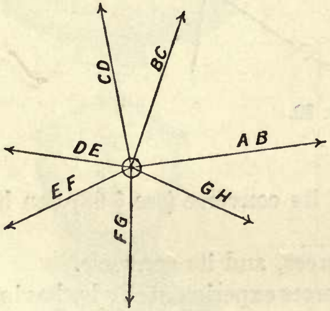


FIG. 80.

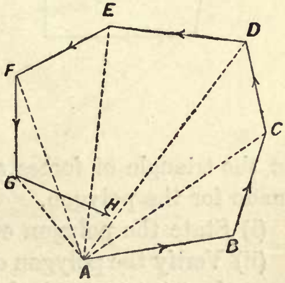


FIG. 81.

Now let us consider the case of a large number of forces acting at a point, as in Fig. 80. We proceed to represent them consecutively by the lines AB, BC, CD, &c. (Fig. 81).

Now from what has gone before, it is plain that *AC* represents the resultant of *AB*, *BC*, and that *AD* represents the resultant of *AC*, *CD*, and therefore of *AB*, *BC*, *CD*. Continuing this argument, we see that *AH*, which we have not drawn in the figure, represents the resultant of all the forces, and that if we were to apply to the point in Fig. 80 a force proportional to *HA*, it would keep all the rest in equilibrium.

AH, then, is the resultant, and *HA* the equilibrant of the forces (or vectors) represented by the sides of the open polygon *ABCDEFGH*.

It follows that if the polygon is *closed*, as in Fig. 82, the forces must be in equilibrium, and statements corresponding to the triangle of forces and its converse (see § 68) can be made for the polygon.

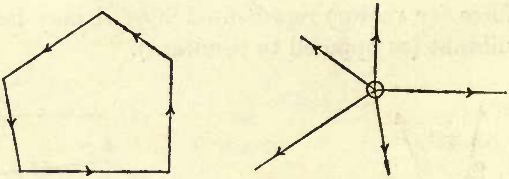


FIG. 82.

(i) State the polygon of forces, and its converse.

(ii) Verify the polygon of forces experimentally by having several strings attached to a ring, as in Fig. 58, and passing them over pulleys, with weights attached to the ends (see Fig. 83). Mark on the paper fastened to the drawing-board the directions and magnitudes of the forces, and then draw the polygon, taking the sides proportional and parallel to the forces in order. If this is done accurately the polygon will be *closed.*

Or the ring may be held by a nail in any other position than that in which the forces are in equilibrium. The polygon will then be *open*, and a force proportional and parallel to the closing line can be applied by passing one more string with the proper weight attached over another pulley clamped in the proper position. Then on removing the nail from the ring the forces ought once more to be in equilibrium.

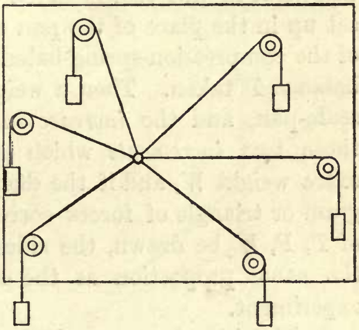


Fig. 83.

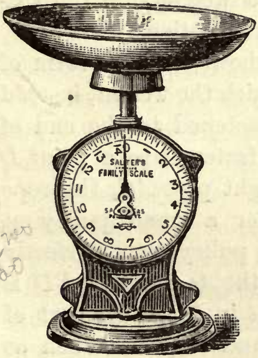


Fig. 84.

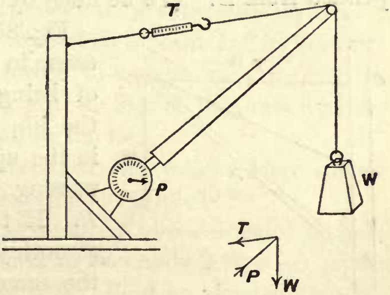


Fig. 84 A.

§ **71. Practical Applications of the Triangle of Forces.** As in some cases it is necessary to measure *pressures* as well as tensions, it is useful to have *compression-spring-balances.*

A drawing of one of these is shown in Fig. 84. This may be used to measure the compressing force in a strut, as, for instance, in Fig. 84 A, which shows a model of a crane. The strut in this case is called the *jib*. The jib should be set up in the place of the pan of Fig. 84, and the readings of the compression-spring-balance P and the tension-spring-balance T taken. Then a weight W should be put in the scale-pan, and the *increase* in P and in T noted. It is these two *increments* which go towards supporting the extra weight W, and if the diagram either of the parallelogram or triangle of forces corresponding to the directions of T, P, W be drawn, the sides should be very closely in the same proportion as the known values obtained by experiment.

(i) The jib of a crane AB is 20 ft. long. The end of it B is fastened by a tie BC 18 ft. long to a point C vertically above A, the distance CA being 9 ft. Find the additional push in AB and pull in BC due to a weight of 1 ton suspended from B. (To be done by construction.)

Fig. 85 shows another form of crane in which the weight, instead of being fastened to the end of the jib, is fastened to a point D in the upright post AC, the rope passing over a small pulley at B. If the pulley runs perfectly smoothly, the tension in BD is the same as in the other part of the rope, and may be taken as equal to W.

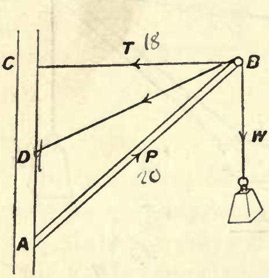


Fig. 85.

In making our polygon of forces, we draw first ab parallel and proportional to W. Then bc may be drawn parallel to BD and equal to ab, since the two forces are equal.

Then remembering that the directions of the forces must be circuital, i.e. the arrows must follow each other all the way round, we can complete the polygon by drawing parallels to the remaining two forces through the points *a* and *c*, either as in Fig. 86 (i) or Fig. 86 (ii).

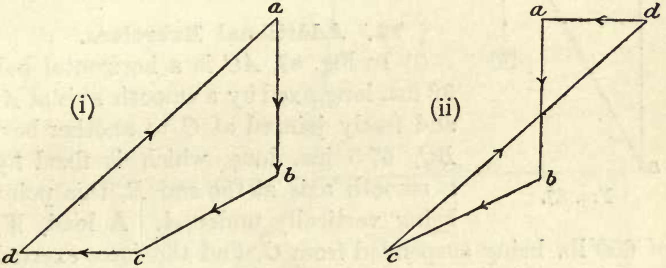


Fig. 86.

(ii) Take as an example a crane with jib *AB* 25 ft. long, attached by a tie *BC* 20 ft. long, *C* being 12 ft. above *A*. A weight of 2 tons is attached to a rope passing over a smooth pulley at *B* and fastened to a point *D* 6 ft. below *C*.

Draw the polygon of forces to a scale of 2 ins. to the ton, both as in Fig. 86 (i), and in Fig. 86 (ii), and find the push in the jib and the pull in the tie.

How would these be altered if the weight were simply attached to *B*, as in the previous example?

(iii) A weight of 20 lb. is hung by two strings of lengths 2 ft. and 1 ft. 6 ins. fastened to two nails 2 ft. 6 ins. apart and in the same horizontal line. Find by construction the tensions in the strings. Verify by calculation, and by actual experiment, using spring-balances to measure the tensions.

(iv) A weight of 1 cwt. is hung from the top of a capital A formed by two struts each 5 ft. long with their feet 6 ft.

apart. Find by construction and calculation the thrust in each strut. If two compression-spring-balances in struts of equal length are available, this may be verified by experiment, taking distances in the same proportion as above.

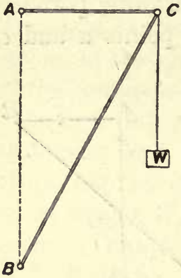


FIG. 87.

§ 72. Additional Exercises.

(i) In Fig. 87, AC is a horizontal bar 32 ins. long fixed by a smooth axis at A, and freely jointed at C to another bar, BC, 67·5 ins. long, which is fixed by a smooth axis at the end B, this point being vertically under A. A load, W, of 600 lb. being suspended from C, find the force exerted in each bar, the weights of the bars being negligible. Your solution may be graphical or by calculation. (*Army.*)

(ii) A weight of 16 lb. is supported on a smooth inclined plane by a force acting horizontally. What is the magnitude of this force if the length of the plane be 10 ft. and its height 6 ft.? (*Navy.*)

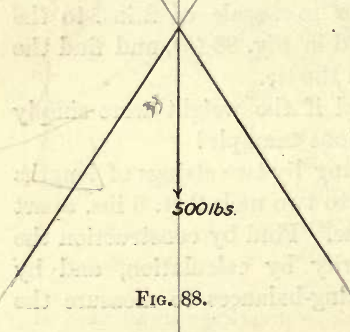


FIG. 88.

(iii) Fig. 88 shows a weight of 500 lb. supported by two equally inclined poles. Find the thrust on each pole. (*Army.*)

(iv) The three forces shown in Fig. 89 act on the point A, which is free to move in any direction. Find what force must be added to the group to keep the point at rest. Write down the magnitude of the force and the angle it makes with the force AB. (*Army.*)

(v) Forces of 1, 2, 3, 4, 5 lb. act on one point in directions parallel to sides of a pentagon taken in order. Find the magnitude and direction of their resultant by a graphical construction.

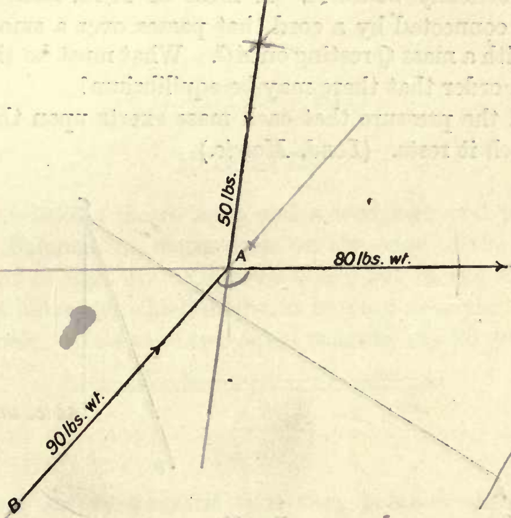


FIG. 89.

(vi) A body of given weight W lb. rests on a smooth plane of inclination a to the horizontal under a force acting up the plane. Find the magnitude of the force and the pressure exerted by the plane on the body.

If the body is a uniform cube with four of its edges horizontal and one face in contact with the plane, and if the force applied to support it acts in a line of greatest slope in that face, find the least inclination of the plane for which the cube will turn about its lowest edge. (*Army.*)

(vii) AB and AC are two smooth planes of lengths 15 ft.

and 10 ft. respectively, sloping downwards in opposite directions from A. B and C are on the same horizontal line 5 ft. vertically below A. A mass of 10 lb. rests on AB, and is connected by a cord that passes over a smooth peg at A with a mass Q resting on AC. What must be the mass of Q in order that there may be equilibrium?

Find the pressure that each mass exerts upon the plane on which it rests. (*Lond. Matric.*)

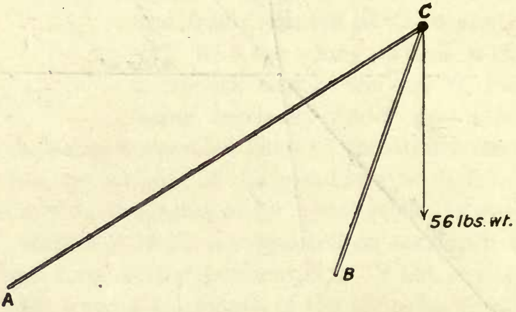


FIG. 90.

(viii) A bar AC (Fig. 90), carrying a load of 56 lb. weight at the end, is hinged to the support A. A prop BC is applied in the position shown, and hinged at B and C. Find the respective loads on AC and CB. (*Army.*)

CHAPTER VIII

MOMENTS

§ **73.** Take a metre scale and a wedge-shaped piece of wood. Balance the metre scale on the edge of the wedge as shown in Fig. 91. Observe the point in the scale at which it balances, which ought to be very near the middle of the scale. Now take two equal weights, say 20 grammes each, and satisfy yourself that they balance each other when placed on the metre scale at equal distances from the balancing-point.

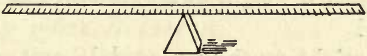


FIG. 91.

Now take weights in the ratio of 2 to 1, say 100 grammes and 50 grammes, and find at what distance the 50 grammes must be placed to balance the 100 grammes at a distance of 20 cms. from the balancing-point. Take weights in any other ratio, 3 to 1, 2 to 3, 3 to 4, and find the ratio of the distances at which they balance. Is there any connexion between the ratio of the distances and the ratio of the weights?

Place any two weights on the scale so as to balance each other. Now double the distance of one of the weights, and double the other weight without altering its distance. You will find that you still have a balance. The two weights tend to turn the scale in opposite directions, and the *turning* effect can be doubled either by doubling the distance or by doubling the force. Let us take 1 gramme weight at a distance of 1 cm. from the turning-point as our *unit* of turning effect. What will be the measure of, i.e. how many units of turning effect will be produced by, a 20-gramme weight at a distance of 25 cms.?

The turning effect of a force about a point is called its **moment.**

Verify for yourself the law that when two forces balance each other about a point, the measures of their moments are the same, e.g. 20 grammes at 25 cms. balance, 50 grammes at 10 cms., the moment of each being 500 units.

(i) At what distance must a weight of 15 grammes be placed to balance a weight of 20 grammes 12 cms. from the turning-point?

(ii) What weight must be placed 18 cms. off to balance a weight of 30 grammes at a distance of 15 cms., all distances being measured from the turning-point?

(iii) Find the weight of a penny using only a 10-gramme weight to balance it.

(iv) On one side put a 50-gramme weight at a distance of 20 cms. from the turning-point. On the other put a 20-gramme weight 30 cms. from the turning-point. Do they balance? Find where a 10-gramme weight must be placed so that it may create equilibrium. Verify the fact that the moment of the 50-gramme weight is equal to the sum of the moments of the other two weights.

§ **74.** In the following experiments a graduated bar with holes at equal intervals, or one made of soft wood, into which small picture-rings can be screwed at different places, will be found useful.

(i) Hang the bar from a nail at one end so that it can swing freely. Then fasten two strings at different points, and fix two pulleys so that the strings may run horizontally over the pulleys and have weights fastened to their ends. Adjust the weights so that the bar still hangs vertically, and verify the fact that

$W_1 \times AB = W_2 \times AC$ (see Fig. 92).

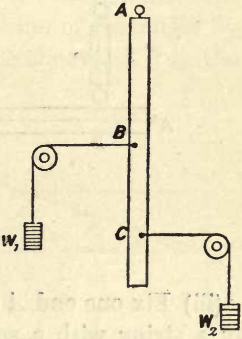


FIG. 92.

Find the point G about which the bar balances. It should be near the middle point if the bar is of uniform thickness. Fix the point A so that the bar can turn freely round A, and fasten a spring-balance to any point B near A, and lift the bar into a horizontal position (Fig. 93). Read the pull P, and note the distances AB, AG. Assume that

$$P \times AB = W \times AG,$$

and hence find W, the weight of the bar. Verify by weighing it in the ordinary way.

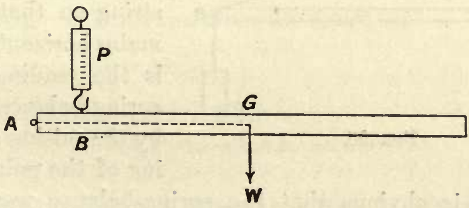


FIG. 93.

(ii) Now hang a weight W' from the end C of the bar, and find what pull on the spring-balance at B is needed to lift the bar into the horizontal position. Verify the fact that

$$P \times AB = W \times AG + W' \times AC \text{ (see Fig. 94).}$$

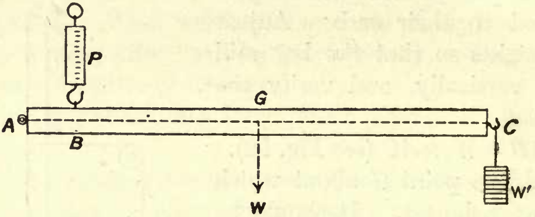


FIG. 94.

(iii) Fix one end *A* of the bar and fasten the other end *B* by a string with a spring-balance to a point *C* vertically above *A* (Fig. 95). Note the reading of the spring-balance, and then hang a weight *W* from *B*, and again note the reading. Is the increase in the reading more or less than *W*? Alter the height of *C*, lengthening or shortening the string so that *AB* remains horizontal. How is the reading of the spring-balance affected by the raising or lowering of the point *C*?

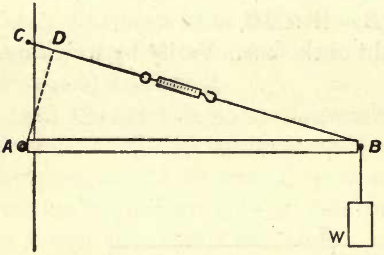


FIG. 95.

(iv) It is obvious that the spring-balance would have least strain on it if it were pulling directly upwards, in which case the additional pull on it due to *W* would be simply equal to *W* itself. Try this, holding the spring-balance in your hand. But if the spring-balance is pulling like *P* in

Fig. 96, part of its force is merely forcing the bar against the fastening at A. Suppose that P makes an angle a with AB. Resolve P into two components as shown in the figure. What is the connexion between P and W? Verify this connexion by experiment, measuring the sine of the angle by finding the lengths AB, BC, and remember that P in the figure means, not the whole pull on the spring-balance, but only the additional pull necessary to support W.

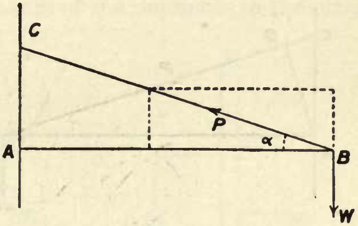


FIG. 96.

Now we must look at the thing from another point of view. The moment of W about A is $W \times AB$. But the moment of P is not $P \times AB$, because P is not taking full advantage of the length AB. To take full advantage of the distance AB, P would have to act at right angles to it; and by pulling obliquely P wastes part of its force. Its turning effect is merely $P \sin a \times AB$; in other words P's distance from A, the turning-point, is no more than the perpendicular AD (Fig. 95). It is obvious from the figure that

$$\frac{AD}{AB} = \sin a$$

$$\therefore \quad P \times AD = P \times AB \sin a$$

$$= P \sin a \times AB, \text{ as above.}$$

§ **75. Definition of a Moment. How is it measured?** *The moment of a force about a point is measured by the product of the force and the perpendicular distance of the point from the line of action of the force.*

Thus if a piece of cardboard be pinned to the table at A

and a string fastened to the cardboard at C be pulled with a force P, the turning effect, or moment, will be measured by $P \times AB$, where AB is the perpendicular from A on the line of action of the force P (Fig. 98).

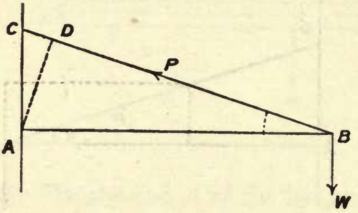


Fig. 97.

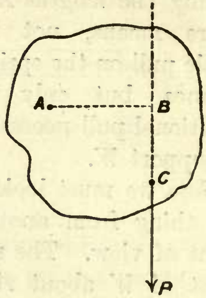


Fig. 98.

§ **76. Conditions of Equilibrium.** Take a set of measurements to verify the fact that the moment of P (Fig. 97) is equal to the moment of W. Measure AB and W in inches and pounds weight, or centimetres and grammes weight. $W \times AB$ remains the same all through the experiment. Remember that the weight of the bar itself is not being taken into consideration, and therefore P in every case must be the *increase* in the pull on the spring-balance caused by W. Measure this increase in several different positions, the perpendicular AD being also measured. Tabulate your results, and you should find that $P \times AD$, measured of course in the same units as $W \times AB$, is the same within from 1 to 10 per cent., the percentage of error depending on the degree of accuracy with which you can read your spring-balance.

(i) Hang the bar from one end as in § 74 (i), and arrange the pulleys as shown in Fig. 99.

Verify the fact that there is equilibrium when the moment of W_1 is equal and opposite to the moment of W_2. We may make use of the same apparatus to verify the fact that when one force pulling one way round balances two or more forces pulling the other way round, the moment in one direction is equal and opposite to the sum of the moments in the other direction. Fig. 100 gives an idea of the arrangement, which is left as an exercise for the student.

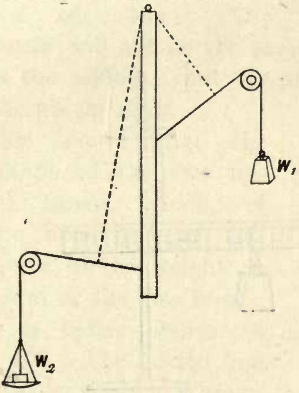


Fig. 99.

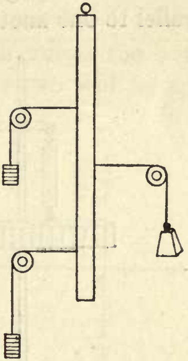


Fig. 100.

If we take one direction of turning, say the opposite way to that in which the hands of a watch move, as positive, the other direction may be taken as negative, and moments may be reckoned + or − according as whether they tend to produce *counter-clockwise* or *clockwise* turning. If we agree to this convention, it is easy to see that, *in cases of equilibrium of forces tending to turn a body round a point, the sum of their moments must be zero.*

§ **77. Parallel Forces. Moments.** We have examined the case of two forces acting at a single point, and we know how to find their resultant. But in practice we are not concerned with single points, and even the smallest object is capable of being turned round as well as of being moved out of its place. The atoms, or molecules, of which a body is composed each has its own weight pulling it towards the centre of the earth, so that the weight of the body is the resultant of a large number of forces which are practically parallel to one another.

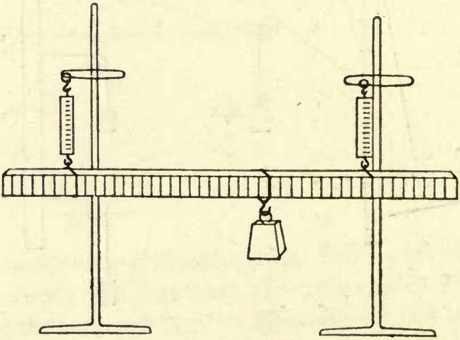


Fig. 101.

We may assume that the weight of a body is the sum of the weights of all of its parts, and we should naturally expect the resultant of two parallel forces to be equal to the sum of the forces. This assumption we may verify by experiment.

(i) To verify that the resultant of two parallel forces in the same sense is equal to the sum of the forces.

Hang up a graduated bar by two spring-balances as shown in Fig. 101. Two retort-stands may be used, as it is easy to shift their positions. The spring-balances should both hang vertically. A better and simpler method is shown in Fig. 102. The bar rests on two compression-spring-balances, which might carry wedges so as to give definite points of support. Then hang on a weight at any point in the bar between the two balances. The increase in the readings of the balances will not be the same for each unless the weight is in the middle; but the sum of the two will be equal to the weight itself.

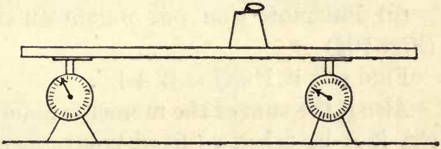


Fig. 102.

This shows that the resultant of the two upward forces, which, of course, must be equal and opposite to the weight, is the sum of the two forces. This is better shown in Fig. 103. The dotted line R must be equal and opposite to W, and it is found by experiment that $P+Q=W$. Hence it follows that $R=P+Q$.

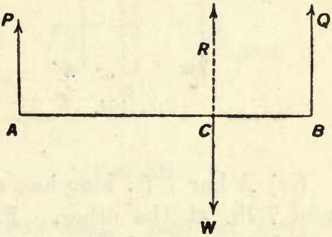


Fig. 103.

Next it will be found that $P\times AC=Q\times BC$, thus verifying the law of moments, taking C as the fixed point.

Suppose A is taken as the fixed point. Then $W\times AC$ should be equal to $Q\times AB$. But $W=P+Q$.

$$\therefore \quad (P+Q)\times AC=Q\times AB$$

$$\begin{aligned}\therefore \quad P\times AC &= Q\times AB-Q\times AC\\ &= Q\times(AB-AC)\\ &= Q\times BC.\end{aligned}$$

Similarly it will be found by experiment that if B be taken as the fixed point, $P \times AB = W \times BC$.

Show that this gives the same result.

(ii) Put more than one weight on the bar, as W and W' (Fig. 104).

Find out if $P + Q = W + W'$.

Also if the sum of the moments about any point vanishes; e.g. if A be taken as fixed point, then we should find that

$$W \times AC + W' \times AD - Q \times AB = 0.$$

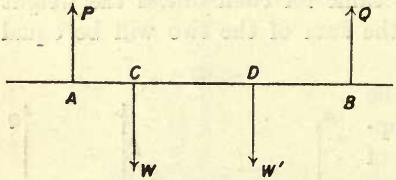


FIG. 104.

(iii) What is the pressure on the wedge in Fig. 91 when the weights are 15 lb. and 25 lb., the weight of the bar being neglected? If the bar weighed $\frac{1}{2}$ lb., what would be the total pressure on the wedge?

(iv) A bar 5 ft. long has a weight of 3 lb. at one end and 7 lb. at the other. Find the point about which it will balance, neglecting the weight of the bar. (Take the balancing-point as x ft. from one end, and write down the equation of moments. This will give x.)

(v) A bar 2 ft. long is supported at its ends, and carries a load of 20 lb. at a point 9 ins. from one end. Find the pressures on the two supports, neglecting the weight of the bar.

(vi) Show by a figure how a weight of 1 lb. may be made to balance a weight of 10 lb. (Principle of the lever.)

(vii) Fig. 105 shows a diagram in section of a wheel and axle. A man winds up the weight W by pushing the handle A of the wheel. If the distance CA is 2 ft. 6 ins., and the radius of the axle *BC* is 5 ins., find the least force with which *A* must be pushed to lift a weight *W* of 3 cwt., and state the direction in which the force must act.

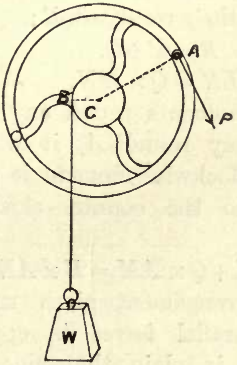


Fig. 105.

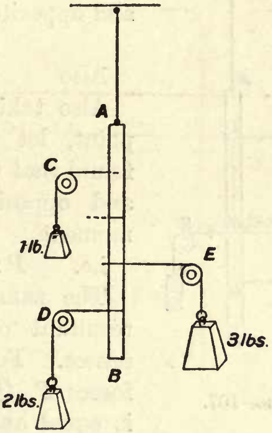


Fig. 106.

§ **78. Parallel Forces** (*continued*). Another method of finding the resultant of two parallel forces in the same sense is shown in Fig. 106. Hang the bar *AB* by a long string attached to *A*. Hang any weights over the pulleys *C* and *D* by strings leaving the bar horizontally. To balance these two weights, *and also to keep the bar vertical,* it will be necessary to hang over the pulley *E* a weight equal to the sum of the other two, and to place *E* at such a level that the vertical distance between *C* and *D* is divided in the inverse ratio of the weights hanging from them.

AB should be hung in front of a vertical board, and the pulleys should run on French nails or bradawls which can be inserted at any required point of the board.

Fig. 107 shows the same thing more generally.

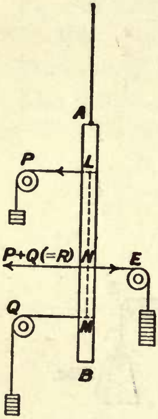


FIG. 107.

In the position of equilibrium for the bar AB it will be found that $E=P+Q$, and E, being the equilibrant of P and Q, must be equal and opposite to their resultant R;

$$\therefore \quad R=P+Q.$$

Also $\qquad P \times LN = Q \times MN.$

Also taking moments round any other point, let us say round A, it will be found that the clockwise moment is equal and opposite to the counter-clockwise moment,

i.e. $\qquad P \times AL + Q \times AM = E \times AN.$

The same arrangement gives us the resultant of parallel forces in opposite senses. For it is plain that since the forces P, Q, E are in equilibrium, each is equal and opposite to the resultant of the other pair; e.g. Q is equal and opposite to the resultant of E and P which are in opposite senses. Now $E=P+Q$, $\therefore\ Q=E-P$. Also it will be found that if we take M as our turning-point, then

$$E \times MN = P \times LM.$$

The results of this experiment may be summed up as follows :—

$R=$ resultant. $\qquad E=$ equilibrant.

(*a*) Parallel forces P and Q in the same sense.

$$R=E=P+Q$$
$$Px=Qy \text{ (Fig. 108).}$$

(*b*) Parallel forces P and Q in opposite senses ($P>Q$).

$$R=E=P-Q$$
$$Px=Qy \text{ (Fig. 109).}$$

In both cases $\frac{P}{Q} = \frac{y}{x}$; in other words the line of action of the resultant divides the distance between the forces internally or externally in the inverse ratio of the forces.

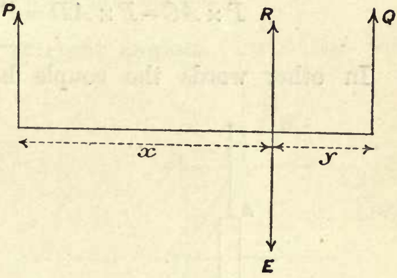


FIG. 108.

(i) Suspend the bar as before, and hang on weights of 2 lb. and 3 lb. in opposite senses at points 6 ins. apart. Find by calculation the magnitude and line of action of their resultant, and verify by hanging on a third weight equal and opposite to this resultant as the equilibrant.

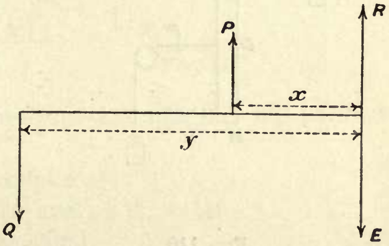


FIG. 109.

§ 79. Couples. When the forces are equal and in the same sense, their resultant is halfway between them.

When the two forces are equal but in opposite senses, their resultant is $P - Q = O$, but since $Px = Qy$, and the resultant falls *externally*, it is plain that its line of action must be so far off that x and y are practically equal; in other words, it must be at an infinite distance. Try the bar at different levels, and it will be found that no single force can be found to balance them. Two such equal forces form a **couple**.

Taking moments round A, we find that the clockwise moment is

$$\begin{aligned} P\times AC - P\times AD &= P\,(AC - AD) \\ &= -P\times CD. \end{aligned}$$

In other words the couple has a counter-clockwise moment round A, measured by one of the forces multiplied by the distance between them which is called the **arm** of the couple. Similarly it will be found that the moment is the same round any other point, such as B or G.

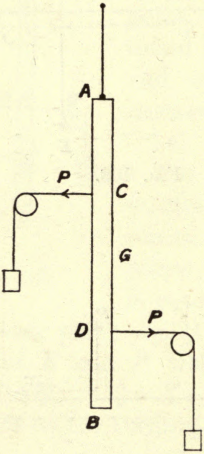


Fig. 110.

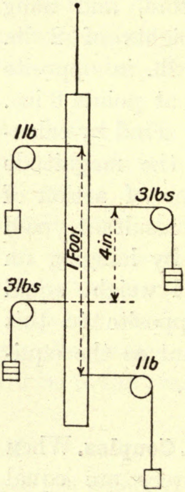


Fig. 111.

A couple can only be balanced by another couple having the same moment, and this couple can act anywhere in the plane.

(i) Balance a couple in which the forces are each 1 lb. and the arm 1 ft. by another in which the forces are each 3 lb. and the arm 4 ins. Fig. 111 shows the arrangement. The second couple may be placed anywhere so long as it is in the same plane as the first couple.

§ **80. Two Couples at right angles.** Pivot the bar *AB* at its middle point *C*, so that it will rest in any position. If the upright board mentioned in § 61 is divided into squares, or covered with squared paper so as to have vertical and horizontal lines, much time will be saved. Then arrange a vertical couple W_1, W_1, and a horizontal couple W_2, W_2, as shown in Fig. 112.

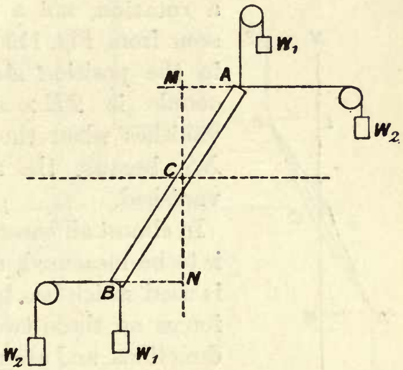


Fig. 112.

The moments of the couples are

$$2\,W_1 \times AM \quad \text{and} \quad 2\,W_2 \times CM.$$

These will be found to be equal.

Hence $$W_2 \times CM = W_1 \times AM,$$

$$\therefore \quad \frac{W_2}{W_1} = \frac{AM}{CM} = \tan ACM.$$

This should be verified with different weights and angles. It is an important result in electrical and magnetic measurements.

§ **81. Magnetic Couples.** Float a magnetized needle on water. It sets north and south; the north-seeking pole of the needle being attracted to the north. But the needle does not sail off to the north, because there is an equal force pulling its south-seeking pole towards the south. The magnetic forces form a couple, of which the effect is to turn the needle into a certain direction, but not to move its centre, in other words, the action of a couple is to cause a **rotation**, not a **translation**. It will be seen from Fig. 113 that when the needle is in the position AB, the moment of the couple is $2E \times AL$, and this moment vanishes when the needle is in the line NS, because the arm of the couple has vanished.

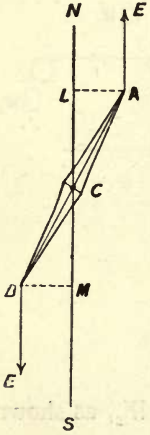


Fig. 113.

In almost all cases where a magnetic force is to be measured, a small magnetic needle is used which has two opposite poles. The forces on these two poles are in opposite directions, and are practically equal to one another, thus forming a couple. The action of this couple is to pull the needle into a certain direction, as in Fig. 51, in which the two equal forces pull directly against one another. When we know the strength of one of these two equal forces, we know what is called the *strength of the magnetic field* at that place. If the field alters rapidly in strength as we go from one place to another, then the force on one end of a small magnet may be noticeably greater than on the other end, in which case the small magnet will be *translated* as well as rotated. This happens near one of the poles of a powerful magnet. Iron filings scattered on paper laid above the magnet are attracted bodily to the poles when quite near to them, whereas when at a moderate distance they merely assume certain definite directions (see Ashford's *Electricity and Magnetism*, § 15).

§ **82. Two Magnetic Fields at right angles.** In Fig. 114 *AB* is a magnet acted on by the earth's force which would

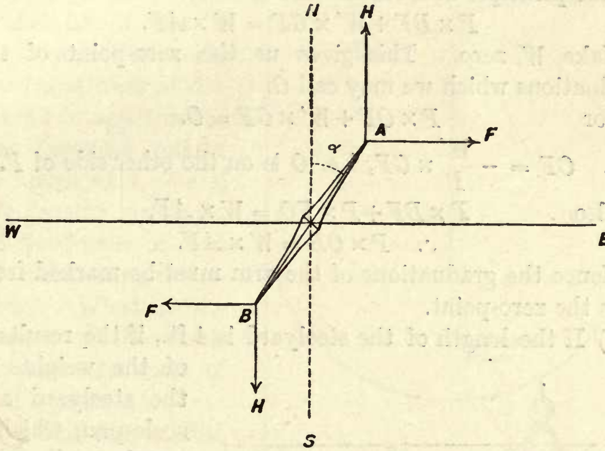


Fig. 114.

pull it into the direction *NS*, but for another magnetic couple *F*, *F* which would pull it into the direction *EW*. Between them they pull it into the position shown in the figure, making an angle α with the line *NS*. Show that

$$\frac{F}{H} = \tan \alpha$$

(*a*) by equating moments, as in § 80.

(*b*) by assuming that the moment of the resultant couple is zero, i.e. that the resultant of *F* and *H* acts in the line *AB*. Use the parallelogram of forces.

§ **83. The Common Steelyard.** The steelyard is a lever used as a balance, and having a graduated arm on which one moveable weight can be adjusted (Fig. 115).

It has a fulcrum at F, a scale-pan at A, and a moveable weight P. If G be the centre of gravity (see § 85) of the bar and scale-pan, and if W' be the weight of these, we have by the principle of moments that

$$P \times BF + W' \times GF = W \times AF.$$

Make W zero. This gives us the zero-point of the graduations which we may call O.

For $$P \times OF + W' \times GF = O.$$

$\therefore\ OF = -\dfrac{W'}{P} \times GF$, i.e. O is on the other side of F.

Also $$P \times BF + P \times FO = W \times AF.$$

$$\therefore\ P \times OB = W \times AF.$$

Hence the graduations of the arm must be marked from O as the zero-point.

(i) If the length of the steelyard is 4 ft., if the resultant of the weights of the steelyard and scale-pan, which is equal to 3 lb., acts in a vertical line distant 16 ins. from the end to which the scale-pan is attached, and if 19 lb. is the greatest weight which can be measured when the moveable weight is 3 lb., find the position of the fulcrum.

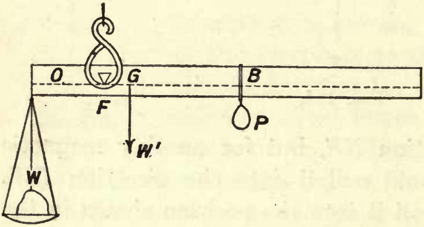


Fig. 115.

§ **84. Exercises.** (i) What are the forces acting upon

(*a*) an oar in rowing?

(*b*) a corkscrew in penetrating a cork?

(ii) Define the moment of a force with respect to a point.

A force of 10 lb. acts tangentially at the circumference of a pulley 2 ft. in diameter. What force must be applied to a rope coiled round the shaft to which the pulley is keyed (or fixed) to prevent the pulley from turning? The diameter of the shaft is 3 ins. (*Army.*)

(iii) Find the magnitude and position of the resultant of parallel forces of 1, 2, 3, 4, 5 and 6 lb. acting in vertical lines at distances 1 ft. apart in the same vertical plane, the forces of 1, 3 and 5 lb. acting upwards, and the forces of 2, 4 and 6 lb. acting downwards. What is the distance of the resultant from the force of 1 lb.? (*Army.*)

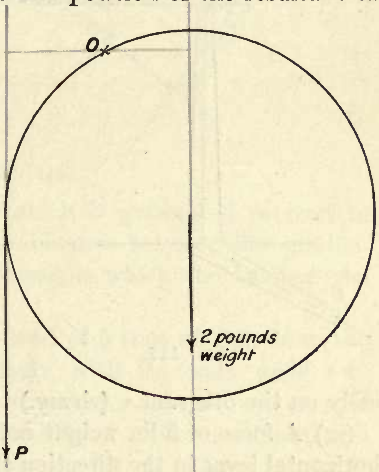


FIG. 116.

(iv) The disc in Fig. 116 weighs 2 lb., and turns about the point *O*. What force *P*, acting in the position shown, is required to hold the disc in the position shown?

(v) Two men carry a weight of 160 lb. slung on a pole (the weight of which may be neglected) at $\frac{1}{4}$ the length of the pole from one end. If the men are at the ends of the pole, what weight will each support? (*Navy.*)

(vi) A wheel and axle is used to raise water from a well. The radius of the axle is 6 ins., and that of the wheel (to which the handle is fixed) is 2 ft. How many turns of the handle will be needed to raise the bucket 50 ft., and what force must be exerted on the handle if the bucket full of water weighs 45 lb? (The circumference of a circle is 3·14 times the diameter.) (*Army.*)

(vii) In a given balance the left arm is 1·01 times longer than the right. Find the true weight of an object which when placed in the left pan appears to weigh 434·80 grammes. (*Lond. Matric.*)

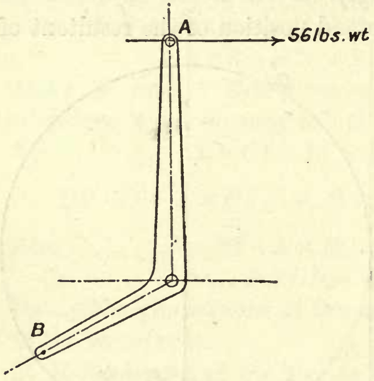


Fig. 117.

(viii) A force of 56 lb. weight is applied horizontally to the long arm of the lever shown in Fig. 117. Find what force would be required to balance it acting vertically at *B*. Find the resultant of the two forces graphically on the diagram. (*Army.*)

(ix) A force of 3 lb. weight acts at the end of a straight horizontal lever in the direction specified by the dimensions shown in Fig. 118 (which is drawn to scale). What force must be applied vertically at *A* to balance it? Use graphical or arithmetical methods as you prefer. (*Army.*)

(x) A beam is placed horizontally across a trestle and let go. What determines whether it will fall, and which way it will fall? (*Navy.*) (See § 85.)

(xi) At one end of a uniform rod 18 ins. long and weighing 1 lb., a weight of 10 lb. is hung. At what point must the rod be supported that it may rest horizontally?

(xii) A uniform steelyard is 2 ft. long and weighs 10 lb.

The pan weighs 35 lb., and is hung from one end. The fulcrum is 3 ins. distant from that end. A weight of 5 lb. slides on the beam, which is graduated to read to quarter-pounds. Find the distance between the graduations, and also the greatest weight which the balance can weigh.

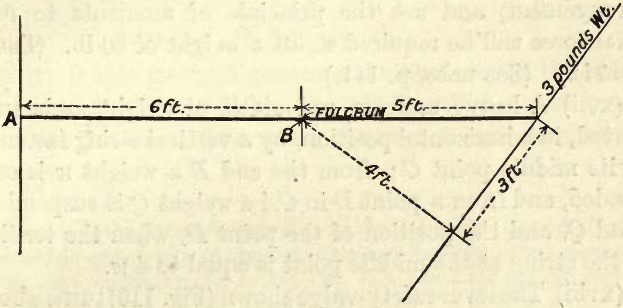


FIG. 118.

(xiii) A beam carrying a load of 5 tons at 3 ft. from the left-hand support rests freely with its ends upon two supports 20 ft. apart. Calculate the pressure on each of the supports. The beam itself is of uniform section, and weighs 2 tons.

(xiv) Two parallel forces whose magnitudes are respectively 10 and 17 lb. weight act in the same plane and in opposite directions, in lines 10 ft. apart. Find the magnitude and position of their resultant, stating the position with reference to the larger force.

(xv) *AB* is a uniform bar 8 ft. long, having a mass of 150 lb. supported in a horizontal position on two vertical props at *A* and *B*. How near an end may a load of 200 lb. be placed if the pressure on a prop may not exceed 250 lb.?

(xvi) In a differential wheel and axle, the diameters of the wheel and the two parts of the axle are 10 ins., 4 ins., and 3 ins. respectively. Draw a diagram of the arrangement, and use the principle of moments to find what force will be required to lift a weight of 60 lb. (*Lond. Matric.*) (See note, p. 141.)

(xvii) A heavy uniform rod *ACB*, of weight *w*, is supported, in a horizontal position, by a vertical string fastened to its middle point *C*; from the end *B* a weight *w* is suspended, and from a point *D* in *CA* a weight *Q* is suspended. Find *Q*, and the position of the point *D*, when the tension of the string at the middle point is equal to 4 *w*.

(xviii) The lever safety-valve shown (Fig. 119) turns about the fulcrum *A*, the resultant thrust of the steam on the valve acts vertically upwards at the point *B*, and the valve is held down by a weight applied at *C*. If *C* is 20 ins. from *B*, and *B* is 3 ins. from *A*, calculate the weight which must be applied so that the valve may just blow, when the upward thrust on the valve is 700 lb. weight. (*Army.*)

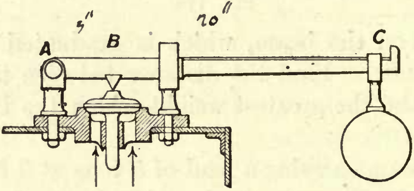


FIG. 119.

(xix) A ditch is bridged over by a light plank *AB*, supported at its two ends. A man of weight *W* stands at any point *P* of the plank.

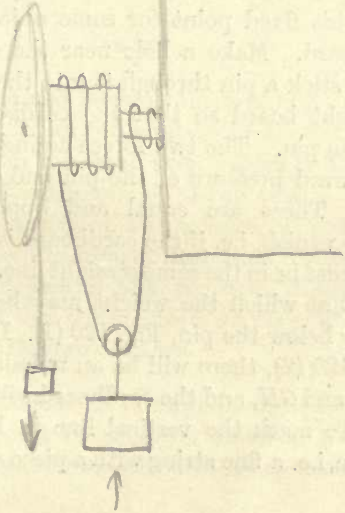
(*a*) Find what portions of his weight are supported by the reactions at the two ends *A* and *B* respectively.

(*b*) Take any point *Q* between *P* and *B*, and prove that the moment about Q of the forces acting on either one of the portions AQ or QB is

$$W \cdot \frac{AP \cdot QB}{AB}.$$

(*c*) If this moment measures the tendency of the plank to break at Q, prove that the plank will be most likely to break in the middle when the man is standing there. (*Army.*)

Note.—In the differential wheel and axle, the weight, instead of being fastened direct to a rope passing round the axle, is attached to a moveable pulley, like the lower pulley in Fig. 178. The axle itself has two slightly different diameters, like the upper block in Fig. 179, and the rope which passes round the lower pulley unwinds itself from the smaller part of the axle while it is wound up on the larger part. Consequently at each complete revolution the weight is raised through a distance equal to half the difference of the two circumferences. The power is applied to a wheel of large diameter, as in the ordinary wheel and axle.



ae C.G. of a body is the point at which the body's
ght may be supposed to act.

e centre of a system of parallel forces is that point through
resultant always passes no matter how the common
ection of the forces is changed provided their magnitud
oints of application remain unchanged

CHAPTER IX

P.T.O.

CENTRE OF GRAVITY

§ 85. Centre of Parallel Forces. The weight of a body is made up of the weight of all its parts, so that it may be regarded as the resultant of a number of parallel forces (see p. 130). We have to show by experiment that this resultant passes through a fixed point, fixed that is to say relatively to the body; so that when the body is turned round, and the weight has a different direction, it still acts through the same point in the body. This will be understood better if we make an experimental determination of this fixed point for some object like a piece of stiff cardboard. Make a hole near the edge of the cardboard, and stick a pin through it into the edge of the table or an upright board so that the cardboard swings quite freely on the pin. The two forces acting on the cardboard are the upward pressure of the pin and the weight of the cardboard. These are equal and opposite, and if the couple is to vanish, i.e. if the cardboard is to stop swinging, the forces must be in the same straight line. This will happen if the point at which the weight may be supposed to act is vertically below the pin, Fig. 120 (1). If the cardboard is as in Fig. 120 (2), there will be an unbalanced couple with moment *W* and *GH*, and the cardboard will be turning round the pin. To mark the vertical line in Fig. 120 (1), hang a plumb-line, i.e. a fine string with a piece of lead at the end,

number of parallel forces acting on the particles of the bo
ch force being proportional to the mass of the particle a
it
entre of this system is the CG of the body.

against the edge of the pin, and make a mark on the lower edge of the cardboard to show where the string hangs, making the mark on the same side of the string as the pin is. Then unpin the cardboard and rule the line joining this mark to the pin-hole. If there is a centre of parallel forces it must be somewhere along this line. Now make a pin-hole in a different place, and repeat the process.

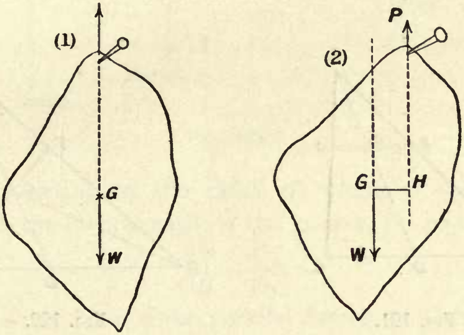


Fig. 120.

Another line is obtained crossing the first line. The centre must be at the point of intersection of these lines. Now make a third line in the same way. Does it pass through the same point of intersection? If so this is the centre. Try whether the cardboard will balance on your finger at this point. If the experiment has been made carefully, the cardboard will rest in any position, if supported by a pin through it at this point, which we may call its **Centre of Gravity**, or its **Centre of Mass.**

§ **86. Special Cases.** It is easy to see where this point should be for bodies of uniform material and uniform shape. A square piece of cardboard, for instance, balances about the point of intersection of its two diagonals. Similarly for a parallelogram. A uniform rod balances about its middle point. A triangular piece of cardboard may be regarded as made up of a number of thin strips, as shown in Fig. 121. The centre of gravity of each of these strips $b\,c$ lies at its middle point d, so that the centre of gravity of the whole triangle must lie in the line joining these middle points, i.e. the median AD. The three medians AD, BE, CF (Fig. 122) intersect at one point G, which is therefore the centre of gravity. It can easily be seen, by examining the similar triangles ABG, DEG, that AG is double of GD, and therefore the centre of gravity of a triangle is one-third of the way up one of its medians. Cut out a triangular piece of cardboard. Bisect the sides, and join the middle points to the opposite corners. Does the cardboard balance at the point of intersection of these lines?

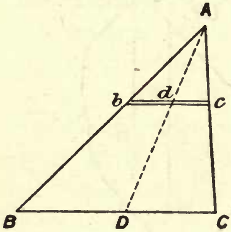


FIG. 121.

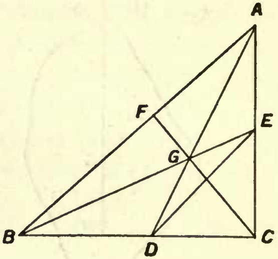


FIG. 122.

§ **87. Centre of Gravity of two or more heavy particles.** The weights of bodies are proportional to their masses (see § 47). Let the force of gravity be g dynes for every gramme. Then the weight of a body whose mass is m grammes is mg dynes. Two masses m_1 and m_2 (see Fig. 123) are acted on by their weights m_1g and m_2g, which are parallel forces. The resultant of these forces is a force

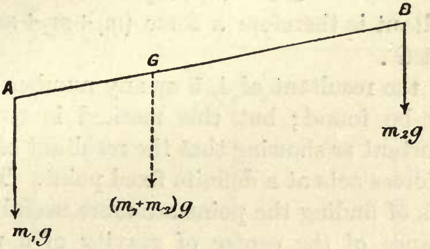


FIG. 123.

$(m_1+m_2)\,g$ acting at the point G, where G divides the line AB in the inverse ratio of the masses, i.e. so that

$$\frac{AG}{GB}=\frac{m_2}{m_1}.$$

Again, if we have three parallel forces, or three masses m_1, m_2, m_3 at the points A, B, C (Fig. 124), the weights of

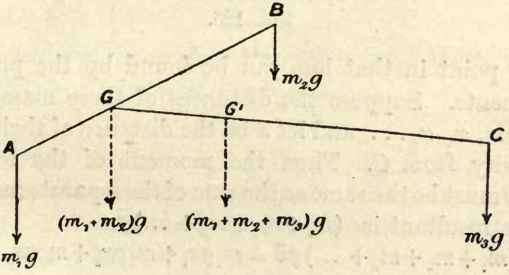


FIG. 124.

the two masses m_1, m_2 can be replaced by their resultant weight acting at G. Then the resultant of all three is found by dividing the line CG in the inverse ratio of the forces at C and G, viz. so that

$$\frac{GG'}{G'C} = \frac{m_3}{m_1 + m_2}.$$

The resultant is therefore a force $(m_1+m_2+m_3)g$ acting at the point G'.

Similarly the resultant of 4, 5 or any number of parallel forces may be found; but this method is tedious, and chiefly important as showing that the resultant of a number of parallel forces acts at a definite fixed point. The following methods of finding the point are more useful:—

The distance of the centre of gravity of a number of masses m_1, m_2, m_3, &c., in a straight line (Fig. 125) from

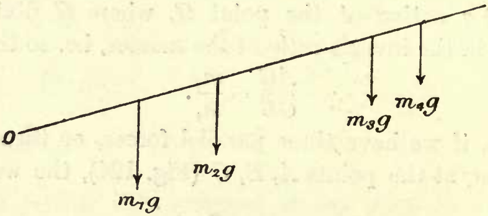


Fig. 125.

a fixed point in that line can be found by the principles of moments. Suppose the distances of these masses from O to be x_1, x_2, x_3 . . ., and let $\bar{x}$ be the distance of their centre of gravity from O. Then the moment of the resultant about O must be the same as the sum of the separate moments. But the resultant is $(m_1+m_2+m_3+ \ldots)g$.

$$\therefore \quad (m_1+m_2+m_3+ \ldots)g\bar{x} = m_1gx_1 + m_2gx_2 + m_3gx_3 + \ldots$$

$$\therefore \quad \bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

$$= \frac{\text{sum of moments}}{\text{sum of masses}}.$$

This important result can be extended to a number of masses not in a straight line.

If m_1, m_2, m_3, &c., are masses whose distances from the two fixed straight lines Ox, Oy are (y_1, x_1), (y_2, x_2), &c. (see Fig. 126), and if their centre of gravity be distant $(\bar{y}, \bar{x})$ from the same two straight lines, we have by taking moments round O that

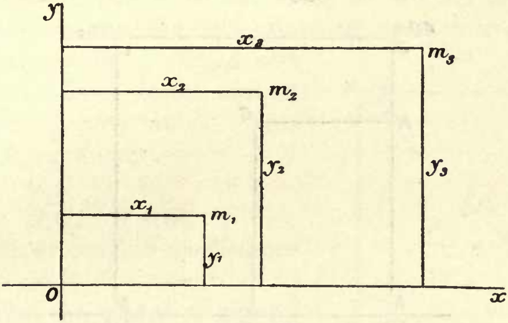


FIG. 126.

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \dots}{m_1 + m_2 + m_3}$$

and by turning the system through a right angle, we get

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \dots}{m_1 + m_2 + m_3 + \dots}.$$

EXAMPLE. Weights of 1 lb., 2 lb., 3 lb., 4 lb. are placed at the corners of a square. Find their centre of gravity. Let a = side of square, G the centre of gravity of the weights (Fig. 127).

$$AM = \frac{1\times 0 + 4\times 0 + 3\times a + 2\times a}{1+4+3+2}$$

$$= \frac{5a}{10} = \frac{a}{2}$$

$$GM = AN = \frac{1\times 0+2\times 0+3\times a+4a}{1+2+3+4}$$
$$= \frac{7a}{10}.$$

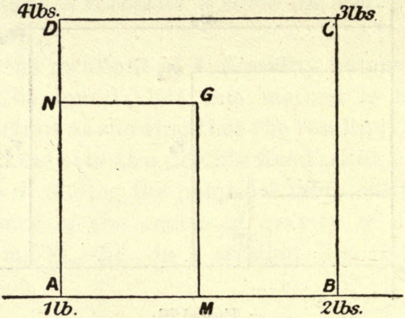

4lbs.
D
3lbs.
C
N
G
A
B
1lb.
M
2lbs.

FIG. 127.

§ **88. Centre of gravity of a 'remainder,' and a 'sum.'** It is sometimes necessary to find the centre of

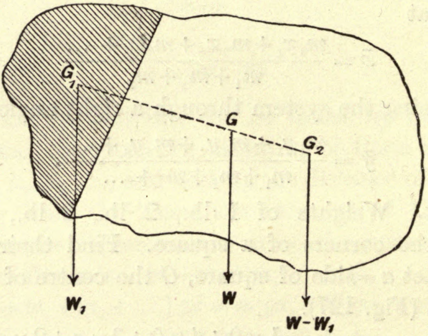

G_1
G
G_2
W_1
W
$W-W_1$

FIG. 128.

gravity of what remains of a figure of known weight and known centre of gravity when part of it, of which the weight and centre of gravity are also known, is removed. In Fig. 128 the weight of the whole body is W acting at G. The weight of the shaded part is W_1 acting at G_1. Then the weight of the remainder must be $W - W_1$ acting at a point G_2 in the line G_1G produced, at such a distance that the two parts, shaded and unshaded, have their mutual centre of gravity at G. This gives the equation

$$\frac{G_2G}{G_1G} = \frac{W_1}{W - W_1}$$

by which we can find the distance G_2G.

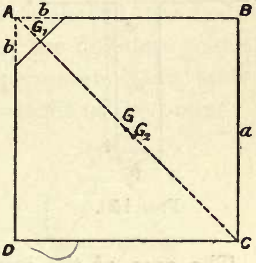


FIG. 129.

(i) Find the centre of gravity of a square with a corner cut off (Fig. 129).

Side of square $= a$; side of triangle $= b$.

Area of triangle $= \frac{1}{2}b^2$; area of square $= a^2$.

The weights are proportional to these areas, and in this way we find that the centre of gravity of the remainder acts at a point G_2 on the diagonal, such that

$$\frac{G_2G}{G_1G} = \frac{\frac{1}{2}b^2}{a^2 - \frac{1}{2}b^2};$$

but $AG = \dfrac{a}{\sqrt{2}}$; $AG_1 = \frac{2}{3}$ of height of triangle

$$= \tfrac{2}{3} \times \frac{b}{\sqrt{2}} = \frac{b\sqrt{2}}{3}.$$

$$\therefore \quad G_1G = \frac{a}{\sqrt{2}} - \frac{b\sqrt{2}}{3}$$

$$= \frac{3a - 2b}{3\sqrt{2}}.$$

$$\therefore \quad G_2G = \frac{b^2}{2a^2 - b^2} \times \frac{3a - 2b}{3\sqrt{2}}.$$

This may be verified experimentally by cutting out a piece of cardboard of the shape shown in Fig. 129, finding its centre of gravity as in § 85, and seeing whether the distance of this point from the centre of the square is the same as that found by substituting the known values of a and b in the above formula.

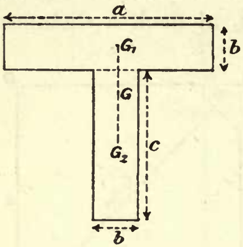


FIG. 130.

(ii) The centre of gravity of a figure made up of two known figures can of course be found by the formula of § 87. We will take as an example the T-shaped figure shown in Fig. 130.

The area of the upright is bc, that of the crosspiece is ab. The centre of gravity of the whole figure is at G,

where $$\frac{GG_1}{GG_2} = \frac{bc}{ab} = \frac{c}{a}.$$

Now $$G_1G_2 = \tfrac{1}{2}c + \tfrac{1}{2}b = \frac{b+c}{2};$$

$\therefore$ G divides this line in the ratio $c : a$.

$$\therefore \quad GG_1 = \frac{c}{a+c} \times \frac{b+c}{2};$$

$$GG_2 = \frac{a}{a+c} \times \frac{b+c}{2}.$$

Verify this by experiment as before.

(iii) Find the centre of gravity of the figure formed by cutting a triangle along the straight line joining the middle points of two sides, and removing the triangular portion.

(iv) A circle of radius a is described, and another circle of radius b touching it internally. Find the centre of gravity of the figure formed by cutting away the smaller from the larger circle.

§ 89. Equilibrium of a body resting on a table. This was alluded to in § 60. The upward pressure of the table on the body is applied at several different points, wherever the two are actually in contact: and the resultant of all these upward pressures must be equal and opposite to the weight of the body acting at its centre of gravity. If the position of this centre of gravity is altered, the body will press and be pressed differently at the different points of contact; but the total upward pressure of the table must always be exactly equal to the weight of the body.

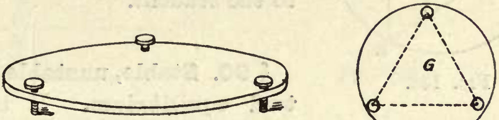


FIG. 131.

Take for example the circular board in Fig. 131, supported on three levelling screws. If the upward pressures of the table on the screws be P_1, P_2, P_3, then $P_1 + P_2 + P_3 = W$, the total weight of the board and the screws and any object placed on the board. The centre of gravity G of the whole must fall inside the triangle formed by the screws, otherwise we shall have an unbalanced couple, and the board will tip over. It is obvious that the resultant of the upward pressures must fall inside the area formed by joining the points at which these pressures act, and if

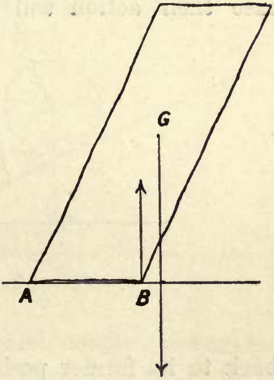


FIG. 132.

the centre of gravity falls outside this area, equilibrium is impossible, as in Fig. 132.

(i) A circular board is supported on three screws which are fixed at equal intervals round the edge of the board, the whole weight being W. Find what is the smallest weight that will tilt the board over, and where it must be placed.

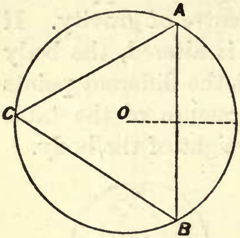


FIG. 133.

Fig. 133 gives a hint for this problem which is left as an exercise to the student.

§ 90. Stable, unstable, and neutral equilibrium. If the forces acting on a body are in equilibrium, it is probable that, when the body is displaced slightly from its position, the forces will be no longer in equilibrium, in which case their action will be either (i) to bring the body back to its former position, or (ii) to take it still further away from that position. An instance of (i) may be found in the pendulum of a clock, and in all cases where a body is suspended with its centre of gravity below the point of

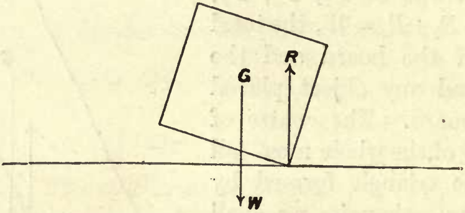


FIG. 134.

support. Equilibrium of this kind is called **stable equilibrium.**

Fig. 134 gives another instance of stable equilibrium disturbed. The block is tilted; but the forces tend to make it return to the position from which it has been displaced.

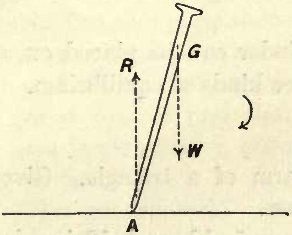


FIG. 135.

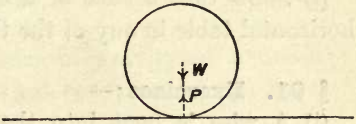


FIG. 136.

An example of case (ii) would be a pin standing on its point (Fig. 135). The centre of gravity *G* of the pin might be vertically above *A*, the point of the pin, in which case there would be equilibrium. But the slightest displacement from this position, in which *W* and *R* are equal and opposite, gives us an unbalanced couple tending to turn the pin over still more.

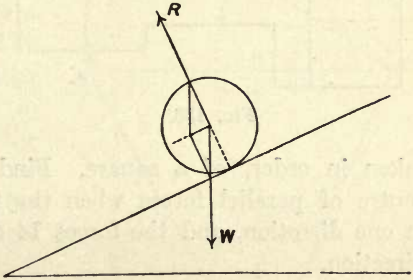


FIG. 137.

There are cases in which the forces remain in equilibrium after displacement, as when a perfectly round uniform ball rests on a horizontal plane. Here the centre of the sphere, which is its centre of gravity, is always vertically above the point of contact (Fig. 136). Equilibrium of this kind is neither stable nor unstable, and is called neutral. Observe that the slightest tilting of the plane destroys equilibrium (Fig. 137), *R* and *W* having a resultant down the plane.

(i) Show that a cone or a cylinder can be placed on a horizontal table in any of the three kinds of equilibrium.

§ **91. Exercises:—**

(i) A wire is bent into the form of a triangle. Give a construction for its centre of mass.

If the sides of the triangle are 5, 12, and 13 ins. in length, find the distances of its centre of mass from the two shorter sides. (*Army.*)

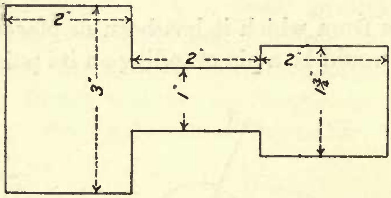


Fig. 138.

(ii) Parallel forces of 10, 18, 14, and 6 lb. weight act at the angular points, taken in order, of a square. Find the position of the centre of parallel forces when the forces 10 and 18 act in one direction, and the forces 14 and 6 in the opposite direction.

(iii) Explain what is meant by the centre of gravity of a body, and state how you would find experimentally the centre of gravity of a wire cage. (*Lond. Matric.*)

(iv) Upon opposite sides of a straight line 8 ins. long, two isosceles triangles are described, one 9 ins. and the other 12 ins. in height. If a thin and uniform sheet of metal were cut so as to have the outline of the resulting four-sided figure, determine the position of its centre of gravity. (*Lond. Matric.*)

(v) A square sheet of cardboard is divided (by a pencil) into four equal squares, and one of them is cut away. Find the centre of gravity of the remaining portion.

(vi) The thin plate shown in Fig. 138 is uniform in thickness and density. Find its centre of gravity, and mark its position carefully on the figure.

Show how to find the centre of gravity of a quadrilateral lamina (*a*) by geometrical construction, (*b*) by experiment.

(The quadrilateral can be divided into two triangles in two different ways. In each case the straight line joining the centres of gravity of the two triangles should be found.)

(vii) From a uniform circular disc of wood 9 inches in radius, a smaller circular disc 6 inches in radius is removed, the centre of the part removed being 3 inches from that of the complete disc. Find the centre of gravity of the remainder. (*Lond. Matric.*)

(viii) In any way you please find the distance of the centre of gravity of the quadrilateral *ABCE* (Fig. 139) from *AC*.

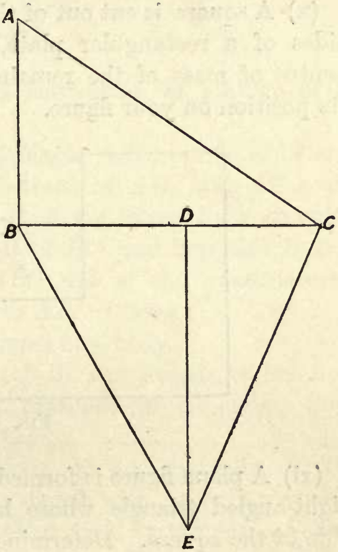


FIG. 139.

(ix) Mark the position of the mass-centre of the system of the three co-planar masses indicated by their mass-centres in Fig. 140. (*Army.*)

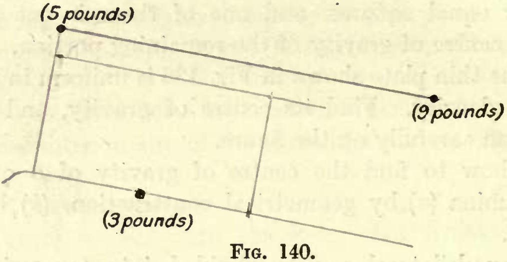


FIG. 140.

(x) A square is cut out of the middle of one of the longer sides of a rectangular plate, as in Fig. 141. Find the centre of mass of the remainder of the plate, and mark its position on your figure.

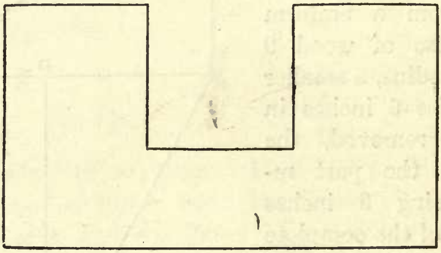


FIG. 141.

(xi) A plane figure is formed of a square and an isosceles right-angled triangle whose hypotenuse coincides with a side of the square. Determine the position of its centre of gravity.

(xii) A rectangular uniform steel plate half an inch thick, with a corner cut off, has the dimensions shown in the diagram (Fig. 142, which is not drawn to scale). Find its mass and the distances of its centre of gravity from *AB* and *AC* respectively. A cubic inch of iron weighs $\frac{1}{4}$ lb.

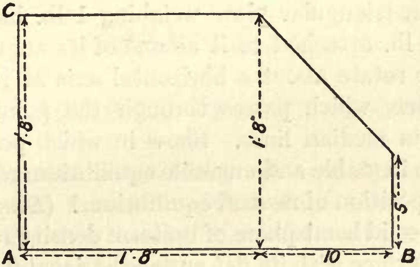


Fig. 142.

(xiii) Find the position of the centre of gravity of a uniform triangular lamina.

In the sides *AB*, *AC* of a triangle *ABC*, points *D*, *E* are taken, such that *AD* is one-third of *AB*, and *AE* one-third of *AC*, and the points *D*, *E* are joined by a straight line; if *F* is the middle point of *BC*, and if *G* and *H* are the centres of gravity of *ABC* and of the quadrilateral *BDEC*, find the ratio of *HG* to *AF*. (*Army.*)

(xiv) Define the centre of mass of a body.

A uniform lever of weight 6 lb. and length 18 ins. has weights of 11 lb. and 7 lb. attached at its ends; find the centre of mass of the system.

(xv) A solid circular cylinder, 18 ins. diameter, 10 ft. long, and weighing 7,000 lb., stands with its axis vertical, and a horizontal force is applied to it at the top. What must be the value of this force, in order that it shall be just able to tilt the cylinder? (*Army.*)

(xvi) When a man balances a stick on the point of his finger, he has to keep the finger moving. Explain fully why this is so.

(xvii) Explain the meaning of the terms *stable, unstable,* and *neutral equilibrium.*

A uniform triangular plate weighing 1 lb. has an extra weight of 1 lb. attached to it at one of its angular points. It is free to rotate about a horizontal axis at right angles to its surface, which passes through the point of intersection of its median lines. Show in which positions the plate will be in stable and unstable equilibrium respectively. Has it any position of neutral equilibrium? (*Lond. Matric.*)

(xviii) A solid hemisphere of uniform density rests upon a horizontal surface with its flat surface horizontal and uppermost. Show that it is in stable equilibrium. (*Lond. Matric.*)

CHAPTER X

CONDITIONS OF EQUILIBRIUM

§ **92. General conditions of equilibrium.** A flat piece of wood of any irregular shape should be taken, and holes should be bored at three different points. Strings should be fastened from these holes to spring-balances, which can be attached to three nails or screws fastened at different points of the floor or table. (If the wall is used, the weight of the piece of wood should be negligible.) The lengths of the strings should be adjusted with two of the spring-balances in position, so that all three can be stretched by simply hooking the third spring-balance on to its nail. See Fig. 143.

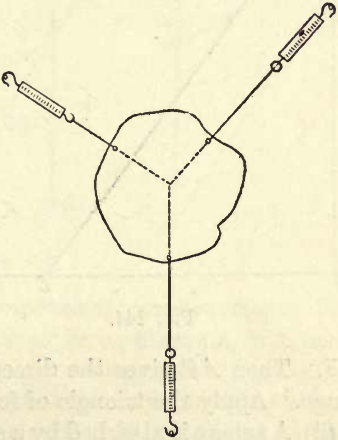


FIG. 143.

Now if the directions of the three strings are marked on a large sheet of paper placed under the wood, it will be found that the lines when produced meet in a point. Incidentally, the triangle of forces (§ 68) may again be verified.

This leads to the following statement :—

If three forces in one plane not parallel to one another keep a body in equilibrium, their lines of action meet in a point, and lines drawn parallel to the forces and not meeting in a point will form a triangle of which the sides are proportional to the forces parallel to them.

(i) The uniform rod *AB* (see Fig. 144) is hinged to a vertical wall at *A*, and supported in a horizontal position by the string *CD* attached to the rod at *C* and the wall at *D*. If the weight of the rod is 10 lb., find graphically the tension in the string, and the magnitude and line of action of the reaction at the hinge. (Bisect *AB* at *E*. Draw *EF* vertical to meet *CD* in *F*. Join *AF*. Then *AF* gives the direction of the reaction of the hinge. Apply the triangle of forces.)

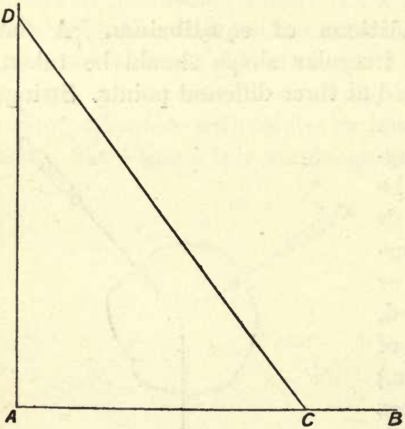


FIG. 144.

(ii) A sphere is attached by a string to a point on a smooth vertical wall, the length of the string being equal to the radius of the sphere. Draw a diagram showing the position of the sphere when in equilibrium, and marking the direction of each of the forces acting on it.

Find the tension of the string if the sphere weighs 10 lb.

(The reaction of the wall passes through the centre of the sphere. So does the weight. Hence the direction of the string must do the same.)

(iii) A rigid uniform bar of weight 10 lb. rests with its lower end in contact with a horizontal rough plane, and makes an angle of 30° with the horizontal, being kept in position by a horizontal thread attached to its upper end. Find the tension of the thread and the magnitude and direction of the force exerted at the lower end.

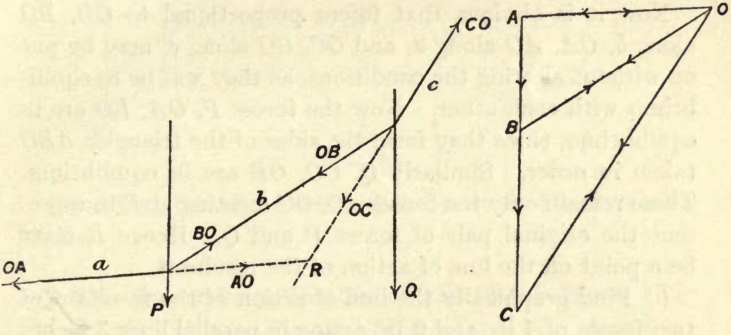


FIG. 145.

§ 93. To find the line of action of the resultant of two parallel forces by geometrical construction. For this we build up a set of forces in equilibrium with each other, which will link the two forces together. Such a set of forces forms a **link polygon** or **link triangle**. Let P and Q be two parallel forces (Fig. 145). It is required to find a point in the line of action of their resultant. Start as if to draw a triangle of forces, of which two sides are to be parallel to P and Q, and proportional to them. Thus AB represents P, and BC represents Q. Then obviously the third side is CA, and a force proportional and parallel to CA will keep P and Q in equilibrium (see § 78). Hence the resultant of *P* and *Q* is another parallel force equal to $P+Q$. Take any point *O* outside *ABC* and join *OA*, *OB*, *OC*. Then forces proportional to *OA*, *AO*, *OB*, *BO*, *OC*, *CO* are obviously in equilibrium. Let us put a link *b* between *P* and *Q* parallel to the line *OB*. (Note that *B* is the point of junction between the *P* and *Q* forces in our 'triangle' *ABC*.) Through the point where this link joins *P* draw a line *a* parallel to *OA*, and similarly, through the point where the link *b* joins *Q*, draw a line *c* parallel to *OC*. Let these two lines meet in *R*.

Now it is obvious that forces proportional to *OB*, *BO* along *b*, *OA*, *AO* along *a*, and *OC*, *CO* along *c*, may be put on without altering the conditions, as they will be in equilibrium with each other. Now the forces *P*, *OA*, *BO* are in equilibrium, since they form the sides of the triangles *ABO* taken in order. Similarly *Q*, *CO*, *OB* are in equilibrium. There remain only the forces *AO*, *OC* meeting at *R* to represent the original pair of forces *P* and *Q*. Hence *R* must be a point on the line of action of the resultant.

(i) Find graphically the line of action of the resultant of two forces of 1 lb. and 2 lb. acting in parallel lines 3 inches apart.

It will be a useful exercise to adapt Fig. 145 to the case of two parallel forces acting in opposite sense.

§ **94. To find the line of action of the resultant of three parallel forces.** Draw the 'polygon' of forces *ABCD*, in which *AB*, *BC*, *CD* represent *P*, *Q*, *R* respectively (Fig. 146). Draw the link *b* between *P* and *Q* parallel to *OB*, *c* parallel to *OC*, and *d* and *a* parallel to *OD* and *OA*, meeting in the point *S*. *S* is a point in the line of action of the resultant.

(i) Find graphically the resultant of parallel forces of

1, 2, 3 lb. acting in lines 2 ins. apart. Verify your result by the method of moments (§ 77).

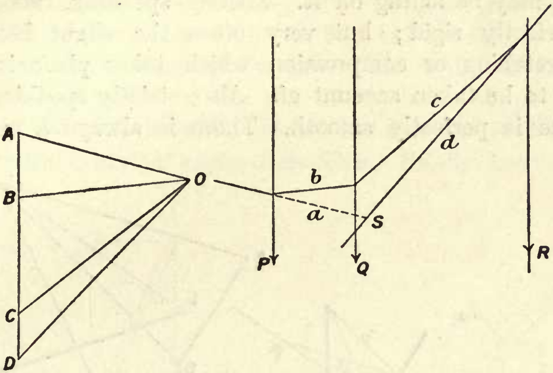


Fig. 146.

(ii) Adapt the above method to the case of four parallel forces.

§ 95. General application of the link polygon. Let P_1, P_2, P_3, P_4, &c., be any forces (Fig. 147). Draw the force polygon *ABCDE*. It is not closed, and *AE* represents the resultant in magnitude and direction (§ 70). Take any point *O* (*O* is called the pole). Join *OA*, *OB*, *OC*, *OD*, *OE*. Then draw a link *b* between P_1 and P_2 parallel to *OB*, *c* between P_2 and P_3 parallel to *OC*, *d* between P_3 and P_4 parallel to *OD*. There remains to draw *a* and *e* from P_1 and P_4 parallel to *OA* and *OE* respectively. The point *R* in the figure where they intersect, is a point on the line of action of the resultant. The student should satisfy himself of the soundness of this construction by putting in arrows with the names of the forces as in Fig. 145.

§ **96. Equilibrium of a rigid body.** By a rigid body we mean a body which keeps its shape, no matter what forces may be acting on it. Strictly speaking, no body is perfectly rigid; but very often the slight bending or stretching or compression which takes place is too small to be taken account of. Also, strictly speaking, no surface is perfectly smooth. There is always a certain amount of friction. But it may be convenient to disregard the friction, and when mathematicians speak of a smooth surface they assume friction to be absent.

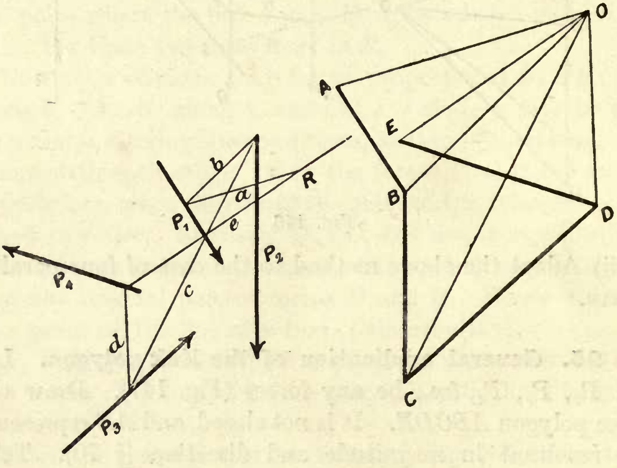


FIG. 147.

When an object is in contact with a smooth surface, the reaction between them must be at right angles to the surface. For if slipping takes place, it must be along the surface, and as there is no friction, there is no force to prevent this slipping; therefore the force at the point of contact must be at right angles to the surface. See Fig. 148, which represents a beam with the top of it in contact with a smooth wall. The reaction of the wall on the beam is at right angles to the wall. In Fig. 149 a beam is shown with its side resting on the top of the wall. If slipping occurred, it would be the side of the beam slipping over a fixed point of the wall. Hence the reaction, if the beam is smooth, is *at right angles to the beam.* Briefly, the reaction is at right angles to the surface which *passes through* the point, i.e. the surface on which slipping takes place.

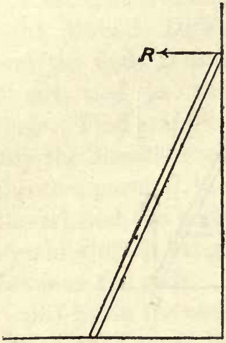


Fig. 148.

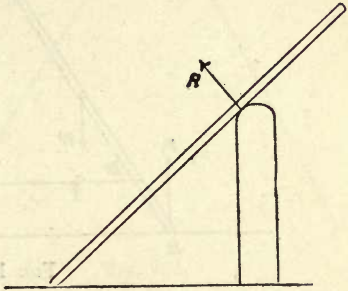


Fig. 149.

If the forces acting on a rigid body keep it in equilibrium, they must balance each other in any direction; otherwise the body would move in that direction. Also their moments round any point must vanish, otherwise the body would turn about that point. An example will help to explain this principle: Two uniform rods AB, AC, of equal length, say 6 ft., each of weight W, are jointed together by a hinge at A, and a string 6 ft. long has its ends fastened to points D, E, 1 ft. from the free end of each rod. The whole is stood up on a smooth floor. Find the tension in the string, the reaction at the hinge, and the pressure on the floor. The forces in a vertical direction must be in equilibrium (§ 78).

$$\therefore\ P_1+P_2-2W = O. \quad \text{Obviously } P_1 = P_2 = W.$$

The forces in a horizontal direction must be in equilibrium. Considering only the rod AB, we have $R-T = O$.

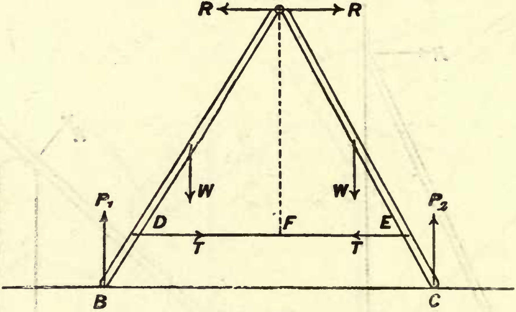


FIG. 150.

The forces tending to turn the rod AB about the point A must be in equilibrium. Therefore their moments about A vanish (§ 76). This gives us

$$P_1 \times \tfrac{1}{2}BC - W \times \tfrac{1}{4}BC - T \times AF = O.$$

Now $BC = \tfrac{6}{5} \times 6$ ft. $= 7{\cdot}2$ ft.

$$AF = \sqrt{5^2-3^2} = \sqrt{16} = 4 \text{ ft.}; \text{ also } P_1 = W.$$

$$\therefore\ W \times 1{\cdot}8 = T \times 4.$$

$$\therefore\ R = T = {\cdot}45\,W.$$

§ 97. Exercises.

(i) A picture frame weighs 8 lb. It is supported by a cord passing over a nail in such a manner that the upper horizontal edge of the frame and the two parts of the cord form an equilateral triangle. Indicate in a diagram the forces acting on the frame, marking the point of application of each, and find the tension in each part of the cord.

(ii) On the two smooth fixed rods *AB* and *AC* (see Fig. 151), equally inclined to the vertical, slide two small rings each of weight 2 oz. Another weight of 10 oz. is knotted at *E* to the light thread *DEF*, and the ends *D* and *F* are tied to the rings. Find graphically the directions of the two parts of the thread, and the position in which it hangs between the rods.

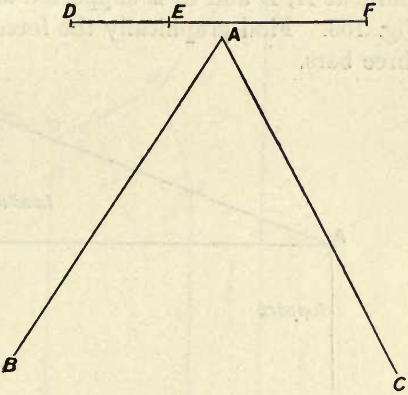


Fig. 151.

(iii) State the condition that must be satisfied if any number of forces acting on a particle hold it at rest.

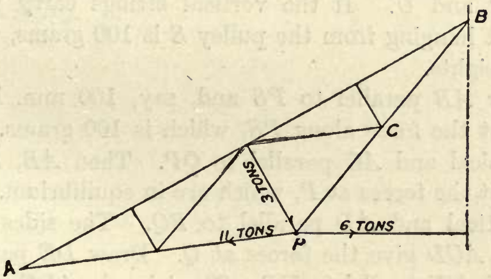


Fig. 152.

Fig. 152 shows a method of supporting a roof. Find the force acting in the tie bar *PC*, the forces in the remaining three bars meeting in the point *P* being given on the drawing.

(iv) A light triangular frame *ABC*, with smooth pin joints at *A*, *B* and *C*, is supported and loaded as shown in Fig. 153. Find graphically the force acting in each of the three bars.

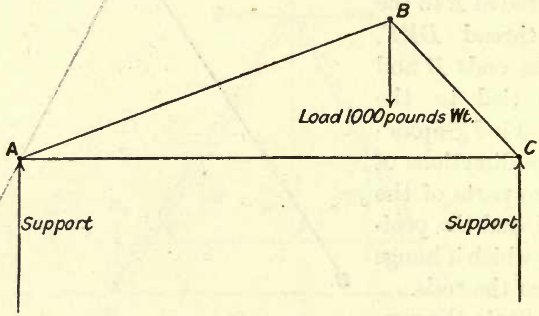


FIG. 153.

(v) Fig. 154 represents a number of strings knotted together at *P*, *Q* and *R*, and passing over smooth pulleys at *S*, *T* and *U*. If the vertical strings carry weights, and that hanging from the pulley *S* is 100 grams, find the other weights.

(Draw *AB* parallel to *PS* and, say, 100 mm. long, to represent the force along *PS*, which is 100 grams. Draw *BC* vertical and *AC* parallel to *QP*. Then *AB*, *BC*, *CA* represent the forces at *P*, which are in equilibrium. Draw *CD* vertical and *AD* parallel to *RQ*. The sides of the triangle *ACD* give the forces at *Q*. Draw *DE* parallel to *RT*, and *AE* parallel to *UR*. The triangle *ADE* gives the forces at *R*.)

(vi) Weights of 1, 2, 3 lb. are placed at the corners of an equilateral triangle. Find by the link polygon the direction of their resultant, and verify by the principle of moments (§ 76).

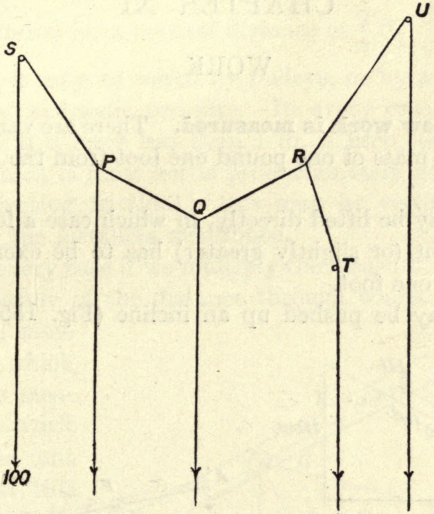


FIG. 154.

orce is said to do work when it moves its point

pplication in the direction of its line of action.

nergy is capacity for work.

wer is rate of doing work.

CHAPTER XI

WORK

§ **98. How work is measured.** There are various ways of raising a mass of one pound one foot from the ground or the table.

(*a*) It may be lifted directly, in which case a force equal to its weight (or slightly greater) has to be exerted for a distance of one foot.

(*b*) It may be pushed up an incline (Fig. 155). If the incline be 1 in x, then the force up the incline necessary to support the weight is $\frac{1}{x}$ lb. (see § 65). This force has to act through a distance of x feet.

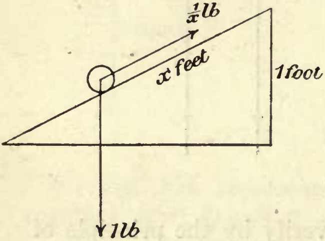


Fig. 155.

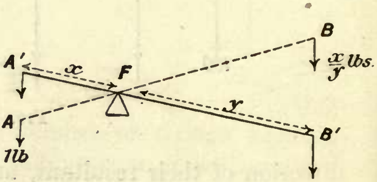


Fig. 156.

(*c*) It may be raised by a lever (Fig. 156). If the arms of the lever are as x and y, then the force at B necessary to balance 1 lb. at A must be $\frac{x}{y}$ lb. by the law of moments (§ 73). Also if A be raised to A', 1 ft. higher than A,

B must be lowered to B', which by 'similar triangles' is $\frac{y}{x}$ ft. lower than B. Hence in this case a force of $\frac{x}{y}$ lb. must be exerted for a vertical distance of $\frac{y}{x}$ ft.

Further, it may be raised by pulleys, or by a wheel and axle, or by hydraulic pressure. In every case it will be found that if a force less than 1 lb. is used, the distance through which it must act is proportionately greater than 1 ft. (neglecting friction); this may be verified by the student in the instances mentioned.

Now in every case if we multiply the measure of the force by the measure of the distance through which it acts, we obtain the same product, 1, which we call the measure of the **work** done. The unit of work in this case is a foot-pound.

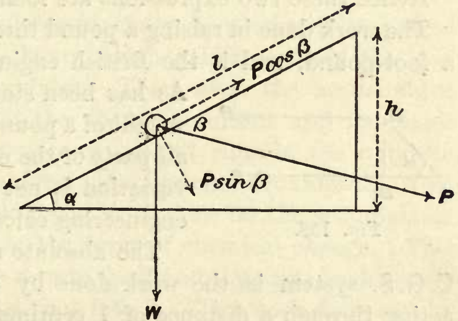


Fig. 157.

Observe that if, with the inclined plane, the force had not been along the slope, but making an angle β with it, we should still have the force $P \cos \beta$ up the plane *doing the work*, while the component $P \sin \beta$ perpendicular to the plane would have no effect in moving the weight. The work done in this case is $P \cos \beta \times l$ (Fig. 157).

$$\text{Now} \quad P \cos \beta = W \sin \alpha = W \times \frac{h}{l}.$$

$\therefore$ Work done $= W \times \frac{h}{l} \times l = Wh$, the same as if the weight had been lifted direct.

Definition. A force is said to do work when it moves its point of application, and the work is measured by the product of the distance moved and the resolved part of the force along that direction, *or* by the product of the force and the projection of the distance moved on the line of action of the force.

Thus in Fig. 158, if the force P makes the point at which it is applied move from A to B, the work done is either

$$P \cos \alpha \times AB \quad \text{or} \quad P \times AD.$$

$$\text{But} \quad AD = AB \cos \alpha.$$

Hence these two expressions are identical.

The work done in raising a pound through a foot is called a foot-pound, and is the British engineer's unit of work.

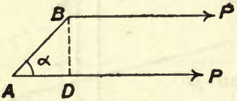


Fig. 158.

As has been stated before (§ 6), the weight of a pound is not the same at all parts of the earth, but the slight variation is not sufficient to affect engineering calculations.

The absolute unit of work on the C. G. S. system is the work done by a force of 1 dyne acting through a distance of 1 centimetre, and this unit is called an erg.

(i) 1 lb. = 453 grammes.
1 gramme weight = 981 dynes.
1 foot = 30·5 cms.

How many ergs in 1 foot-pound?

(ii) A kilogramme is lifted through 1 metre. Express the work done (*a*) in ergs, (*b*) in foot-pounds.

§ **99. Ability to do work. Energy.** When the weights of a ‘grandfather’s’ clock are wound up, work has to be done against gravity: enough work to keep the clock going for 8 days. Then by the escapement action regulated by the pendulum the weights are gradually allowed to descend, and in doing so they keep the clock going, and give back the work that has been put into them. The wound-up weights possess **energy**, which is the power or faculty of doing work. So does the mainspring of a watch that is wound up. The watch is kept going by the spring gradually unwinding itself under the regulation of the balance-spring. The balance-wheel would soon stop swinging if it were not for the impulses given to it through the escapement action by the uncoiling mainspring. When this has uncoiled itself the watch stops.

When an accumulator is charged, a current is passed through it for some hours which oxidizes one of the lead plates of the accumulator. Then when the accumulator is disconnected from the charging current and its plates are connected by a wire, a current runs in the opposite direction and the oxidized plate becomes deoxidized. Here we have a case of electrical work done on the accumulator which is stored up in the form of chemical change. This chemical change can reverse itself and thus give back nearly all of the work that is put into it. The work done can be measured, it being known that a current of 1 ampere at a pressure of 1 volt does 10^7 ergs per second.

In all these cases we have examples of work stored up, and a body possessing the power of doing work by virtue of its position is said to possess **potential energy.** The wound-up weight, the coiled mainspring, the charged accumulator all possess potential energy.

Toy engines are sold which have to be wound up. There is a spring which is wound up and gradually uncoils itself, making the engine run on the floor. Other toys carry a heavy fly-wheel, which can be spun so as to rotate quickly. Then when the toy is placed on the floor it runs until the friction of the various parts brings the fly-wheel to rest. This is an instance of a body possessing energy by virtue of its motion. Energy of this kind is called **kinetic energy**. Fig. 159 shows the connecting rod and crank of an engine. It is obvious that when the crank is in the positions C_1, C_2, the piston cannot do work, for the only direction in which the crank can move is at right angles to the direction of the force. This is known as a 'dead point,' and the heavy fly-wheel of the engine is designed to carry the crank past these dead points by means of its kinetic energy. The kinetic energy of the fly-wheel is created by the piston when it is not at either of the dead points; and the energy of the piston comes ultimately from the potential energy stored up in the coal.

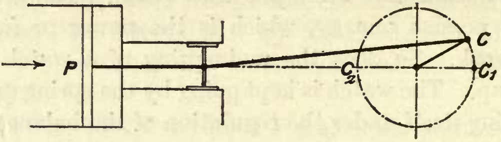


Fig. 159.

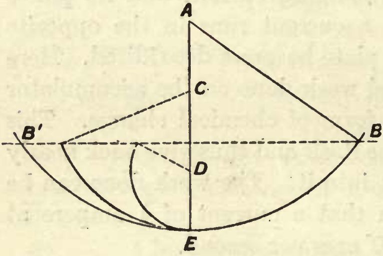


Fig. 160.

(i) To find to what height the kinetic energy of a moving body will carry it. (Galileo's experiment, Fig. 160.)

Suspend a ball by a string from a nail *A*. Draw a horizontal line *BB′* on the wall, and let go the ball from *B*. Does the ball swing as far as *B′*? Now draw a vertical line *AE* through *A*, and put a nail anywhere in this line, at *C* or *D*. The nail should be long enough to catch the string, and so change the centre of the circle described by the ball. Again let go the ball from *B*, and note whether the ball swings as high as the line *BB′*.

What conclusions do you draw from this experiment?

(ii) Why does a pendulum swing? If it swings, why does it ever stop?

(iii) Give any instances you know of **oscillatory** motion.

§ **100. Conservation of energy.** Is the student prepared to accept the statement that for every loss of potential energy there is a corresponding gain of kinetic energy, and vice versa? Even in the swing of a pendulum there is a certain amount lost in each swing. The speed acquired by a cyclist, who mounts at the top of a hill, and coasts down it, is *not* enough to carry him to the same height up the next hill; and with a free wheel the discrepancy between the potential energy at the start and the finish of such an experiment is even more disappointing. The pellets fired straight up into the air do *not* return with the same kinetic energy as was theirs when they left the gun. In all these instances friction and air-resistance interfere with our theory. The great discovery of Dr. Joule, that *energy, like matter, is indestructible*, will be referred to later on (see § 109). For the present it must be enough to state that for every loss by friction there is a corresponding gain of heat energy, and that in cases where friction and air-resistance are not taken into account, **the sum of the kinetic and potential energies remains unaltered.**

§ **101. What happens when work is done on a body that is free to move?** If a stone of weight W is allowed to fall, it acquires, in falling through a distance h, a velocity v given by the equation $v^2 = 2gh$.

This gives us $h = \frac{v^2}{2g}$; and the work done by gravity is equal to Wh.

The stone having acquired this speed possesses a store of energy of its own, and we shall assume that it will require the same amount of work to stop it as has been expended in creating its motion.

To take a simple case. A nail is being hammered into a post. The hammer weighs 1 lb. and hits the head of the nail with a speed of 10 ft. per second. How much work is it capable of doing, and what is the average resistance of the post to the nail, if the blow drives the nail in half an inch? The mass of the nail need not be taken into account.

To measure the work in foot-pounds, we must take the equation $h = \frac{v^2}{2g}$, where h is the height the hammer should have fallen from to acquire 10 ft. per second velocity.

$$h = \tfrac{50}{32} = \tfrac{25}{16} \text{ ft.}$$

$$\therefore \quad \text{work done} = \tfrac{25}{16} \text{ foot-pounds.}$$

This work is converted into motion of the hammer which is brought to rest in a distance of half an inch, i.e. $\frac{1}{24}$ ft.

Hence if R be the average resistance in pounds weight

$$R \times \tfrac{1}{24} = \tfrac{25}{16}.$$

$$\therefore \quad R = \frac{3 \times 25}{2} = 37\tfrac{1}{2} \text{ lb. weight.}$$

§ **102. Work done by gravity on a body falling freely.** If a stone has an initial velocity u, the speed after a fall through a height h is given by the equation $v^2=u^2+2gh$ (see § 19).

If the mass of the stone is m grammes, its weight is mg dynes, where g is measured in centimetre-second units (see § 47). Hence the work done by gravity, in a vertical height of h centimetres, is mgh ergs.

But $$2gh = v^2-u^2.$$

$$\therefore \quad mgh=\tfrac{1}{2}mv^2-\tfrac{1}{2}mu^2.$$

If the stone is thrown directly upwards with a velocity u, the velocity after it has risen through a height h is given by the equation $v^2=u^2-2gh$. In this case gravity diminishes the *kinetic energy*; meanwhile the stone is rising and acquiring *potential energy*. When it reaches its highest point, v is zero, and the kinetic energy is zero; but the potential energy has reached its maximum, and when the stone falls again, the kinetic energy increases as the potential energy diminishes.

§ 103. Work done by gravity on a mass descending a smooth curve. If the mass be m, its weight is mg, and the work done in descending from A to B is obviously $mg \times AC$ (Fig. 161). The resistance of the curve does no work on the mass, as its direction is always at right angles to the direction of the curve, i.e. of the motion of the mass.

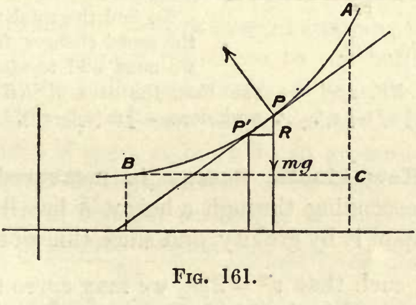


FIG. 161.

Now the work done by gravity between P and P' (supposed very close together) is $mg \times PR = mg \sin PP'R \times PP'$.

Now $mg \sin PP'R$ = force along curve $= m\frac{\delta v}{\delta t}$, where v is the velocity, and δv the change in it between P and P' in the short interval of time δt (see § 20 (vi)). $PP' = \delta s$.

$$\text{Work done between } P \text{ and } P' = m\frac{\delta v}{\delta t} \times PP'$$
$$= m\delta v \cdot \frac{\delta s}{\delta t}$$
$$= mv\delta v, \text{ since } v = \frac{\delta s}{\delta t}, \text{ (see § 18 (iv))}$$

i.e. the work done by gravity in bringing the mass a very small distance along the curve is equal to the momentum and the small change of speed.

Now set off along a straight line a distance OM (Fig. 162) to represent v, and a small additional distance MN to represent δv, and draw a straight line OAB making 45° with the first line. Then draw MP, NQ perpendicular to the first line.

Obviously $MP = OM = v$,
and $NQ = ON = v + \delta v$.

The area of the strip $PMNQ$ is $v\delta v$, and therefore, if we suppose it multiplied by m the mass, it will represent the work done corresponding to a small change in speed δv.

To find the total work done as the speed changes from v_1 to v_2, we must add together the strips from AS to BR, and we see that the area $ASRB$ is equal to $OBR - OAS = \frac{1}{2}v_2^2 - \frac{1}{2}v_1^2$. $\therefore$ work done $= \frac{1}{2}m(v_2^2 - v_1^2)$.

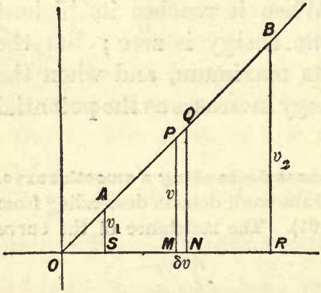


FIG. 162.

§ **104. How kinetic energy is measured.** Since a weight W descending through a height h has Wh units of work done upon it by gravity, and since this work develops a velocity v, such that $v^2 = 2gh$, we may agree to write $\frac{v^2}{2g}$ for h, and call $\frac{1}{2}\frac{Wv^2}{g}$ the **kinetic energy** of the moving mass.

$\frac{\frac{1}{2}mv^2}{32}$ = Kinetic energy = $\frac{mv^2}{2g}$

If the C. G. S. system is used, and the weight W is expressed in dynes, $W = mg$, and the kinetic energy in ergs is $\frac{1}{2}mv^2$.

If, however, we wish to obtain the kinetic energy in foot-pounds and the weight W is given in pounds, then the foot-second value of g must be substituted in the expression $\frac{1}{2}\frac{Wv^2}{g}$, so that the kinetic energy in foot-pounds is

$$\tfrac{1}{2} \times \frac{1}{32 \cdot 18} \times (\text{weight in pounds}) \times (\text{square of velocity in ft. per sec.}).$$

For most purposes $\frac{1}{32}$ is near enough. It is approximately true at all points on the earth that a weight of one pound acting on a mass of 32 lb. would produce in it unit (ft. sec. sec.) acceleration (see § 47).

For example, a bullet of mass 1 oz. and diameter 0·4 in. is moving with a velocity of 1,500 ft. per second. How far will it penetrate a plank of which the average resistance is 50 tons to the square inch? $Ps = \frac{1}{2}$

The equation $\frac{1}{2}mv^2 =$ work done in stopping bullet $= R\,x$, where R is the average resistance to the bullet, and x is the distance the bullet penetrates, is true only when the different units are chosen to correspond. Thus if the C. G. S. system is used, m must be in grammes, v in cms. per second, R in dynes, and x in cms. If m is in pounds and v in feet per second, it is more convenient to put $m = \frac{W}{g} = \frac{\text{pounds weight}}{32}$. Then force may be in the ordinary engineer's unit of pounds (weight) and work in foot-pounds.

The diameter of the bullet being 0·4 ins., the average resistance is $50 \times 2240 \times 3 \cdot 14 \times (0 \cdot 2)^2$ lb.

Hence the equation to find x feet, the distance to which the bullet penetrates the board, is

$$\tfrac{1}{2} \times \tfrac{1}{32} \times \tfrac{1}{16} \times (1500)^2 = 50 \times 2240 \times 3{\cdot}14 \times {\cdot}04 \times x.$$

$$\therefore \quad x = \frac{1500 \times 1500}{2 \times 32 \times 16 \times 112000 \times 3{\cdot}14 \times {\cdot}04}$$

$$= \tfrac{5}{32} \text{ ft.} = 1\tfrac{7}{8} \text{ in.}$$

§ **105. Power.** Suppose that 1,000 bricks weighing 7 lb. each have to be carried up to a height of 40 ft. This means that 280,000 foot-pounds of work have to be done. A child could carry up a brick at a time and so do all the work if only time enough were allowed him. A man who could carry up 20 bricks at a time would do the work much quicker.

Power is measured by rate of doing work.

A horse pulls a load of 1 ton up an incline of 1 in 40 at 5 miles an hour ; at what rate is he working?

Neglecting the work done in overcoming friction, we may assume that the horse is pulling with a force equal to the weight of $\frac{1}{40}$ of a ton (see § 65). In one minute it moves its load through $\dfrac{5 \times 1760 \times 3}{60}$ ft. $= 440$ ft.

$$\therefore \text{ rate of doing work} = \frac{2240}{40} \times 440 = 24640 \text{ foot-pounds}$$

per minute.

When James Watt was estimating the power of his steam-engines, he calculated that the average horse could work at the rate of about 22,000 foot-pounds per minute. He added on 50 per cent. to this and reckoned 33,000 foot-pounds per minute as one horse-power, so that his steam-engines should not suffer by comparison with the average horse. A **horse-power** means 33,000 foot-pounds per minute or 550 foot-pounds per second.

The unit of power on the C. G. S. system is 10^7 ergs per second, and is called a **watt.** It is used in electrical measurements, and is connected with the units of pressure and current by the equation

$$\text{volts} \times \text{amperes} = \text{watts}.$$

Thus a 100-volt lamp taking half an ampere is using 50 watts. Twenty such lamps would be using 1,000 watts, i.e. a **kilowatt.**

(i) What horse-power is necessary to take a gun and carriage weighing 1 ton up a slope of 1 in 10 at the rate of 12 miles an hour, neglecting friction?

The force necessary to pull 1 ton up a slope of 1 in 10 is a little over $\frac{1}{10}$ of a ton, i.e. 2 cwt.; let us take it as 2 cwt.

$$12 \text{ miles an hour} = \frac{12 \times 5280}{60} = 1056 \text{ ft. per minute.}$$

$\therefore$ work must be done at the rate of $2 \times 112 \times 1056$ foot-pounds per minute.

$$\therefore \quad \text{horse-power} = \frac{2 \times 112 \times 32}{1000} = 7{\cdot}168.$$

(ii) A man weighing 10 stone runs upstairs, getting 4 ft. higher every second. At what rate is he doing work?

He must be exerting an average force rather greater than his own weight, because of the upward movement which has to be created; but neglecting this we find that he is doing $10 \times 14 \times 4$ foot-pounds per second. This is rather more than one horse-power, and no man could keep it up for any length of time.

§ **106. Kinetic energy of a fly-wheel.** A wheel with a heavy rim, and with a spindle on which a weight can be wound up, is mounted on ball-bearings. The rim should be smooth so that a strip of paper can be fastened on to it closely. A vibrating steel spring of frequency about 10 per second, carrying a paint-brush, as in the apparatus described in § 42, should be mounted so that its vibrations are across the rim with the paint-brush just in contact with the paper. A cord should be wound on the spindle, to which a weight can be attached (Fig. 163).

The distance the weight descends for one revolution of the wheel can be measured fairly accurately by finding the distance corresponding to several revolutions. Marks should be made on the strip of paper on the rim of the wheel to indicate the position of the paint-brush at the beginning and at the end of the descent of the weight, a distance of, say, one foot. When the weight is in position, the spring should be set in vibration, and the weight allowed to fall. If the cord is put on so as to detach itself after the weight reaches the end of its fall, the wheel will go on with the velocity acquired; and by measuring the wave-lengths on the paper we can calculate this velocity as well as the acceleration during the time the weight was descending. From the final velocity of the fly-wheel we can calculate the velocity v of the weight at the end of its fall.

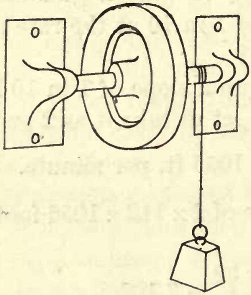


FIG. 163.

The work done by the weight W in descending through a height h is Wh foot-pounds. The kinetic energy of the weight at the end of its fall is $\frac{Wv^2}{2g}$ foot-pounds. This kinetic energy, which is lost on impact against the ground, is less than Wh, and the rest of the energy has been spent in overcoming friction and in giving kinetic energy to the fly-wheel. The energy spent in overcoming friction can be calculated by finding what weight will just keep the fly-wheel in steady motion (tested as in § 43 by the equality of the wave-lengths). Suppose this weight to be w, then the work done against friction is wh foot-pounds. Hence the kinetic energy of the fly-wheel must be

$$Wh - wh - \frac{Wv^2}{2g} \text{ foot-pounds.}$$

By experimenting with various weights, in each case noting the velocity of the rim of the wheel, as well as that of the weight, it will be found that the kinetic energy of the fly-wheel is roughly proportional to the square of the velocity of the rim. The values of the kinetic energy and of the square of the velocity should be plotted on squared paper.

§ **107. Moment of inertia.** It is obvious that the rim of the wheel moves faster than any other part of it; but all parts of it are moving with the same angular velocity, i.e. they all describe 4 right angles (2π radians) in the time of one revolution. If ω stand for this angular velocity, and R for the radius of the rim, n for the number of revolutions per second, then $\frac{\omega}{2\pi} = n$, and the velocity of the rim is $2\pi nR = R\omega$. Hence the kinetic energy of a particle of mass m at a distance r from the centre is $\frac{mr^2\omega^2}{2}$ ergs (§ 104). Now the wheel is made up of many particles at varying distances from the centre, and the total kinetic energy is the sum of all the terms $\frac{mr^2\omega^2}{2}$, which may be written $\sum\frac{mr^2\omega^2}{2} = \frac{\omega^2}{2}\sum mr^2$. (See § 127.)

The quantity Σmr^2, which means the sum of all the little masses ('molecules') of which the wheel is composed, each multiplied by the square of its distance from the centre, is called the **moment of inertia** of the wheel.

If we can find the length of a radius at which the whole mass of the wheel may be supposed to be concentrated, then if M be this whole mass and R this supposed radius, the moment of inertia is MR^2, and R is called **the radius of gyration.**

The moment of inertia of the fly-wheel can be calculated from the foregoing experiments. For a disc of radius r rotating about an axis through its centre and at right angles to its plane, the moment of inertia can be shown by the calculus to be $\frac{Mr^2}{2}$.

§ **108. Exercises**:—

(i) Explain why a pendulum when set in vibration continues to vibrate. Describe the variations of (*a*) velocity, (*b*) energy, of the pendulum during a complete swing.

(ii) If a particle whose mass is 4 ounces has a velocity of 1,000 ft. per second, through what distance will it overcome a force of 360 pounds-weight before coming to rest?

(iii) Distinguish between work and energy.

A weight of 10 kilogrammes falls through 30 metres in three stages, being stopped and started again after each 10 metres of fall. Compare the total amount of energy with that which would have been obtained if the stone had fallen directly through 30 metres.

(iv) If the stone in § 102 has a horizontal velocity u_1, as well as a vertical velocity u, show that the resultant velocity is $\sqrt{u^2+u_1^2}$, and that the change in kinetic energy is still mgh, if the horizontal velocity is unchanged.

(v) A simple pendulum consisting of a weight hanging by a light string from a fixed point swings through an angle of 20° on each side of the vertical. Find the greatest velocity of the weight, the length of the string being 5 ft. (*Army.*)

(vi) Define the terms force and kinetic energy. How is the force acting upon a body related to the kinetic energy it develops in it?

A bullet of mass 2 oz. leaves the muzzle of a rifle with a speed of 1,200 ft. per second. If the barrel is 3 ft. long, find the average value of the force exerted upon the bullet as the result of the explosion of the powder. (*Lond. Matric.*)

(vii) A train is moving at a uniform speed of 40 miles per hour along a level track and the pull exerted upon it by the engine is equal to the weight of two tons. How much work, in foot-pounds, is expended by the engine per minute against frictional resistance to the motion of the train?

What further knowledge would be required in order to calculate the frictional resistance at any instant if the speed of the train were not uniform? (*Lond. Matric.*)

(viii) A train weighing 300 tons starts from rest and is accelerated until its speed is 50 miles an hour. Find the kinetic energy acquired by the train during its acceleration, and express it in foot-tons to two significant figures. (*Army.*)

(ix) Define the terms pound, foot-pound, poundal, and the weight of a pound. Which of these vary, and which do not, at different parts of the earth's surface?

A cage is descending the shaft of a coal-mine with increasing velocity. What would be the result of attempting to weigh a body in the cage (*a*) with an ordinary balance, (*b*) with a spring-balance? (*Lond. Matric.*)

(x) A rifle bullet weighs an ounce and travels 2,500 ft. per second. A block of iron used for driving piles weighs 8 cwt., and just before striking travels 25 feet per second. Which has the greater energy and which the greater momentum?

(xi) A rifle bullet whose mass is 25 grammes leaves the muzzle of a rifle with a velocity of 640 metres per second. Find its kinetic energy, being careful to state the units in which your answer is expressed. (*Lond. Matric.*)

(xii) A body of mass 1 lb. is projected vertically upwards with a velocity of 80 ft. per sec. Find its kinetic energy and its potential energy after 1, 2·5, 5 seconds, respectively, and show that the sum of its kinetic and potential energy is constant until it has struck the ground. (*Army*)

(xiii) A weight of 56 lb. falls vertically 10 ft.; 56 weights each of 1 lb. fall through the same distance. Is the energy expended in each case the same? Give reasons for your answer.

(xiv) How many units of work does a man of 154 lb. do altogether in carrying a hod of mortar weighing 40 lb. up a ladder 30 ft. high? Say what unit you use.

(xv) A mass of 7 lb. is dragged from rest across a level table by a mass of 3 lb. hanging over the edge of the table and attached to the first mass by a long thread. What is the velocity of the second mass after it has fallen 3 ft. if one-tenth of the potential energy transformed has been used against frictional resistance?

Indicate how the result would be affected if the table, instead of being horizontal, were tilted upwards towards the edge over which the string passes. (*Lond. Matric.*)

(xvi) A man winds up a clock in half a minute and in so doing raises the weight 4 ft. In falling the weight drives the clock for a week. Point out the transformations of energy which have taken place in these operations. (*Army.*)

(xvii) A force of 50 poundals acts upon a mass of 700 lb. for 10 seconds. Calculate the momentum possessed by the body at the end of the interval and also its kinetic energy, being careful to write down the units in which the results are expressed.

(xviii) A mass of 1,000 lb. is moving in a straight line at a speed of 30 miles per hour. Its energy is reduced to 500 foot-pounds in 12 seconds. Calculate the retardation and the force which acted to produce it.

(xix) Explain why in catching a cricket ball the hand is drawn back in ' making ' the catch.

A man in catching a cricket ball of mass $5\frac{1}{4}$ oz., travelling with a speed of 64 ft. per second, draws his hand back through 1 ft. Find the average force that the ball exerts on his hand while coming to rest. (*Lond. Matric.*)

(xx) Say whether any work is done in each of the following cases, giving clear reasons for your answers:—(*a*) A pillar supporting a roof; (*b*) a stone at the end of a string being whirled with constant speed in a horizontal circular path; (*c*) an engine drawing a train with constant velocity along a level railway line. (*Lond. Matric.*)

(xxi) A pile weighing 4 cwt. is being driven into the ground by a weight of 3 cwt. let fall on it from a height of 13 ft. If each blow drives the pile $\frac{3}{4}$ of an inch, show that resistance of the earth to penetration is between 13 and 14 tons weight. (Assume this resistance to be a constant force, and that after the blow the weight and the pile move on together till stopped by this force.) (*Army.*)

(xxii) Find the energy stored in a train weighing 250 tons and travelling at 60 miles per hour. How much energy must be added to the train to increase its speed to 65 miles per hour?

(xxiii) A train weighs 200 tons. Find what force must be exerted on the drawbar connecting the train to the engine to draw the train up an incline of 1 in 50 (that is, 1 ft. vertically for 50 ft. horizontally), the resistance to traction on the level being 12 pounds-weight per ton. Calculate the work done (in foot-tons) if the incline is 5 miles long. (*Army.*)

(xxiv) Explain what is meant by the work of a force. What is horse-power?

A given weight W is hoisted up to a given height h by a workman. During the first third of the height the weight is made to move upwards with a constant acceleration, during the second third with a constant velocity, during the last third it is brought gradually to rest with a constant retardation. Supposing that the resistances (including gravity) are represented by a given constant retarding force R, and that the whole time of the ascent is equal to that of falling from rest through a height nh, prove that the difference of the works done by the man in the first and third portions of the ascent is $\frac{25\,Wh}{18\,n}$.

State also the whole work done. (*Army.*)

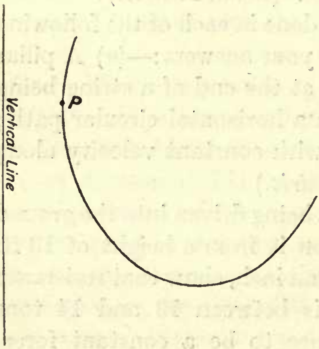


Fig. 164.

(xxv) A ring P is strung on the smooth wire shown by a curve in Fig. 164 and is let go in the position shown. Will it fly off the wire? Give your reasons. (*Army.*)

§ 109. Conservation of energy. Dr. Joule of Manchester made a series of experiments in the years 1843 to 1849, by which he established a numerical relation between the unit of work and the unit of heat. Taking, as the unit of heat, the quantity of heat required to raise 1 lb. of water

1° Fahrenheit, he found that to accomplish this by mechanical means 772·7 foot-pounds of work must be done. The experiments were of three kinds. First, he heated water by churning it with paddles, measuring the rise in temperature in a known mass of water, and the work expended in producing this heat. Then he measured the heat produced by the friction between two metal surfaces, and the work done against friction. Thirdly, he measured the cooling of air which was allowed to expand after being compressed. Since then the work done by an electric current in heating a wire immersed in water has been measured and compared with the heat produced. Callendar's method, by measuring the heat produced by friction of a silk belt with a metal cylinder, is described in § 114. The results of the most recent experiments seem to indicate 778 foot-pounds as the *mechanical equivalent of heat*, i.e. the amount of work equivalent to the heat required to raise 1 lb. of water 1° Fahrenheit.

Joule's work proved what Bacon, Newton, and Boyle had guessed, that heat is a form of motion, or, more accurately, a form of energy. The energy apparently lost in friction is converted into heat, and heat-energy can, under certain conditions, be converted into mechanical work. Energy may take other forms. It may be stored up as potential energy in the clouds, when the sun's heat evaporates the water of the sea, and raises the winds that carry that vapour. 'When the clouds are full of rain, they empty themselves upon the earth,' and the rivers and waterfalls turn our mills. The sun's energy, in the form of heat and light, makes plants and trees to grow. Our coal-beds are storehouses of energy, old forest trees, formed by the chemical activity of sunlight.

By heat or mechanical means we may create an electric current, by which energy may be conveyed to a motor; or by which a wire may be made white hot. These are a few examples of the transformations of which energy is capable. But in all of them we find that *we cannot create energy*, though we may transform it. And although it is equally true that *energy* is *indestructible*, yet it is easily *lost* in the sense of becoming useless for mechanical purposes. The heat caused by the friction of the hub of a wheel, or by the impact of a stone when it strikes the ground, is useless to us. A very large proportion of the heat communicated to the boiler of a steam engine is lost in the sense of not being converted into mechanical work, and all our engines and machines give out less energy than they receive. But still there remains the great law, which must always be associated with Joule's name.

The law of the conservation of energy has been stated thus :—

The total energy of any body or system of bodies is a quantity which can neither be increased nor diminished by any mutual action of these bodies, though it may be transformed into any of the forms of which energy is susceptible.

In the chapters which follow this law will be assumed.

[For Examples on Power, see end of Chapter XII.]

CHAPTER XII

FRICTION

§ **110. Friction. Experiments.** Take a block of wood of a rectangular shape, say 4 × 3 × 2″, fix a staple at one end of it, and hook a spring-balance (reading to $\frac{1}{4}$ oz. or to grammes) on to the staple (Fig. 165). Lay the block on the table (a reasonably smooth table) and find what pull is necessary just to start it moving. Is the same force necessary to keep it moving when once it has started?

Instead of laying the wood on the table, lay it on a sheet of glass (or of paper) fastened on to the table. Again measure the force necessary to start it, and to keep it moving.

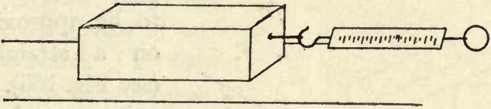


Fig. 165.

Try the same block on other surfaces.

On examining the different faces of your wooden block, you will probably find two faces of different area which are both cut in the direction of the fibre, while the third is cut across the fibre. Choose two surfaces which are alike, and find whether the area of the surface makes any difference to the friction by trying the block with these two surfaces in turn in contact with the table.

Find the weight of the block. Then by loading it with gradually increasing weights find how the forces of friction vary with the load (including the weight of the block). Tabulate your results thus :—

Wood on wood.		
Load.	Limiting Friction.	Sliding Friction.

By **limiting friction** is meant the greatest amount of friction which can be called out ; any force less than this will not start the process of sliding. After it has started, a smaller force will maintain the movement.

If the results obtained are plotted on squared paper, the points will be found to lie approximately on a straight line (see Fig. 166). Some points may be above and some below : but if a thread is stretched tight and laid among the points, it will be easy to determine a position which gives the best average distribution of points on either side of it. When this position has been found, it may be assumed that the deviations of the points from this line are due to errors of observation, and that the line

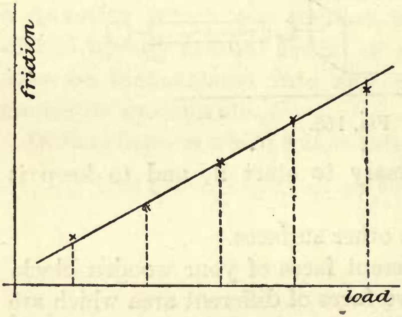


FIG. 166.

itself gives the true connexion between the load and the friction.

§ **111. Coefficient of friction.** In the foregoing experiments you will have come to certain conclusions as to the circumstances which affect the force of friction. Make a statement of the way in which friction is affected, (*a*) by the nature of the surfaces in contact, (*b*) by the area of the surfaces in contact, (*c*) by the pressure between the surfaces.

The gradient of the line in Fig. 166 may be measured by the tangent of the angle which it makes with the horizontal. It is obviously the average ratio of the friction to the load. This ratio is called the **coefficient of friction** for the two surfaces in contact. Find the coefficients of friction of various surfaces, e.g. wood on glass, glass on glass, iron on wood, &c., by the same method.

Your results may be checked by means of an inclined plane. If the two surfaces belong to a plane of which the inclination can be altered, and a block resting on the plane, the plane can be tilted until the block just begins to slip (Fig. 167).

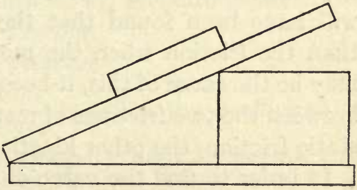


Fig. 167.

By the triangle of forces (Fig. 168) it will be seen that the ratio of the friction F to the pressure between the surfaces P is the same as the ratio of the height of the planes to the base, i.e. tan a, where a is the inclination. a is sometimes called the **angle of friction.** It is obvious that for inclinations less than this limiting angle the whole limiting force of friction is not needed to prevent the block from slipping; and as the tangent of the angle increases from 0 up to its value when slipping begins, so the amount of friction called out increases up to its limiting value. It is *only the limiting friction* which bears any fixed ratio to the pressure; so the value of the coefficient of friction need only be taken into account in cases where the surfaces are slipping or about to slip. For other cases the friction is just as much as is wanted to prevent motion and *no more.*

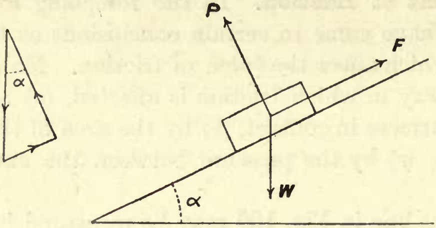


Fig. 168.

§ **112. Limiting friction of rest and of motion.** It will have been found that the *limiting friction* is greater than the friction when the motion has begun. Whatever may be the cause of this, it becomes necessary to distinguish between the two frictions of rest and motion. One is called static friction, the other kinetic friction.

In order to find the value of the kinetic friction between two surfaces, it becomes necessary to arrange that one should slide over the other with uniform velocity. For this purpose the vibrating steel spring described in § 42 will be found useful. The friction board should be smooth and free from knots, and the block which slides on it should be long enough to admit of a strip of paper being pinned on it to receive the wavy curve traced out by the vibrating paint-brush. The force necessary to keep the block in motion (it must be started by hand) can be measured by the weight of the scale-pan and its contents (see Fig. 169). If a difficulty is found in getting uniform motion, it will be better to use a spring-balance, as in Fig. 165, instead of the pulley and weight. A set of readings for different loads may be taken as in § 110, and the results plotted on squared paper as before.

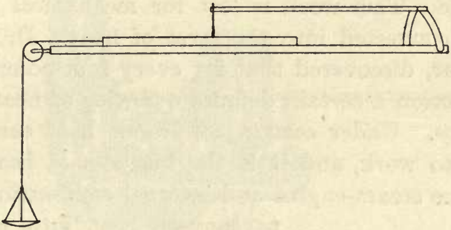


Fig. 169.

Make a table of results showing (*a*) pressure between the surfaces; (*b*) friction; (*c*) $\frac{\text{friction}}{\text{pressure}}$.

As before, the ratio $\frac{\text{friction}}{\text{pressure}}$ is called the coefficient of friction. The symbol μ is generally used to denote this in algebraical language, so that $\frac{F}{P} = \mu$, or $F = \mu P$.

§ **113. Lubricated surfaces.** The friction of two metal surfaces, say, cast-iron on cast-iron, is very different when the surfaces are lubricated from what it is between the same surfaces dry. If possible, a series of observations should be made with two lubricated surfaces to find out how the area of the surfaces in contact and the pressure affect the value of the friction, and whether the coefficient of friction is independent of the load or not.

§ 114. Work done against friction. Mechanical equivalent of heat. If we can measure the force of the friction and the distance through which the rubbing surfaces slide over each other, the product of these two gives the work done. This work is lost for mechanical purposes; but it is converted into the form of **heat.** Dr. Joule, of Manchester, discovered that for every foot-pound of work lost in friction a certain definite quantity of heat made its appearance. Under certain conditions heat can be converted into work, and it is the business of heat engines, such as the steam-engine and internal combustion engines, to convert heat into mechanical work. But there is always a great loss of heat in the most economical of engines. Hence, if we wish to measure **the mechanical equivalent of heat,** it must be by converting work into heat, and not by the reverse process.

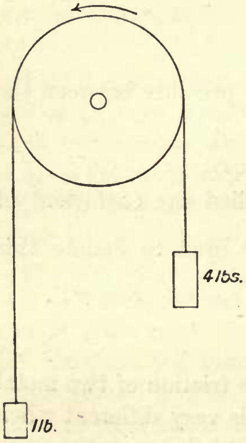


Fig. 170 (i).

The friction of a belt passing over a revolving wheel can be measured if we know the tension of the two parts of the belt where it leaves the wheel. For instance, in Fig. 170 (i), if the wheel, revolving in the direction of the arrow, keeps the two weights hanging from the ends of the strap at rest, it is plain that the force of friction between the wheel and the strap must be equal to the weight of 3 lb. If the circumference of the wheel is 2 ft., then $3 \times 2 = 6$ foot-pounds of work are done against friction at each revolution, and to make this wheel revolve 100 times a minute would require $\frac{600}{33000}$ horse-power.

Callendar's apparatus for measuring the mechanical equivalent of heat is shown in Fig. 170 (ii). Here the wheel is replaced by a revolving brass cylinder containing a measured quantity of water, the temperature of which can be read by a thermometer. The silk belt has a large and a small weight at its ends, and a spring-balance on the side of the smaller weight helps to keep the belt steady when the cylinder is revolving. A ' cyclometer ' records the number

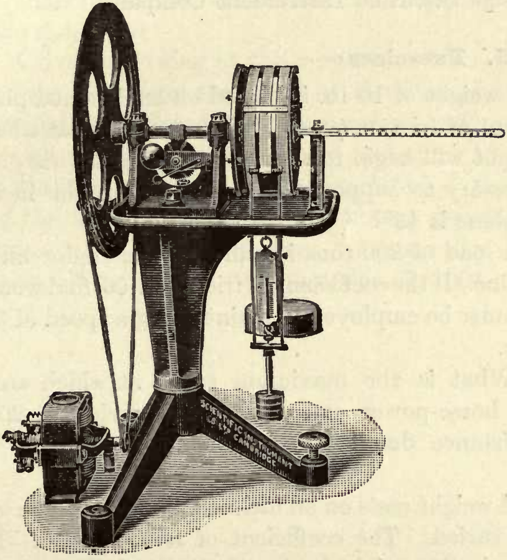


Fig. 170 (ii).

of revolutions. The friction of the belt warms the water and calorimeter, and to avoid errors due to radiation into the room, the water should be at first a few degrees below the temperature of the room, and at the end about the same number of degrees above that temperature. The apparatus, designed by Prof. H. L. Callendar, F.R.S., is made by the Cambridge Scientific Instrument Company.

§ 115. Exercises:—

(i) A weight of 10 lb. is placed on an inclined plane, the coefficient of friction (static) being ·35. Find at what angle the weight will begin to slip. What force up the plane will be necessary to support the weight when the inclination of the plane is 45°?

(ii) A load of 200 tons is pulled by an engine on a horizontal line. If the coefficient of friction is ·06, find what horse-power must be employed in maintaining a speed of 30 miles an hour.

(iii) What is the maximum speed at which an engine of 100 horse-power can pull a train weighing 200 tons, the resistance due to friction being ·7 per cent. of the load?

(iv) A weight rests on an inclined plane the slope of which can be varied. The coefficient of friction is ·6. Find at what angle the weight begins to slip, and, if the friction of motion is $\frac{3}{4}$ of the limiting friction of rest, find the speed acquired in 2 seconds after slipping has begun. How does this compare with the speed acquired, supposing the friction to be completely destroyed?

(v) If the resistance of friction, &c., is 12 lb. weight per ton on level rails, what horse-power will be necessary to drag a train of 250 tons up an incline of 1 in 100 at 15 miles an hour?

(vi) A stone slab lies on a horizontal board, the coefficient of friction between them being 0·23. To what angle (to the nearest degree) can the board be tilted before the stone slips?

(vii) A slab, 5 lb. in weight, rests on a horizontal plane. A horizontal pull of 1 lb. weight would just move it. To what angle (to the nearest degree) can the plane be tilted before the slab slips?

(viii) A river is flowing at three miles an hour. If two steamers are ascending the river at three miles an hour, one propelled by paddle-wheels or a screw, and the other pulling itself up by means of a chain laid along the bed of a river, the former will have to exert twice the horse-power of the latter, although the resistance overcome and the distance travelled in any given time are the same in both cases. Why is this? (*Nature*, Jan. 7, 1904.)

CHAPTER XIII

MACHINES

§ **116. Machines.** A theoretically perfect machine ought to be capable of giving out as much work as is put into it. No machine is perfect, and the ratio of the work put in to the work given out is always what the lamented Euclid used to call a ratio of lesser inequality; in other words, it is less than unity. It is known as the **efficiency** of the machine. The simplest form of machine is perhaps the single fixed pulley, which serves the purpose of changing the direction of a force.

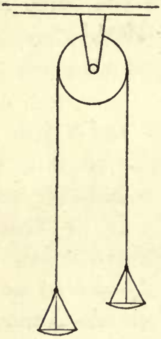


Fig. 171.

Take two scale-pans at the ends of a light string passing over a fixed pulley. Put in equal weights, and then keep adding small weights to one of them until it just begins to move. The difference between the weights represents the force necessary to overcome friction. If the difference is less than this, motion does not begin. The total force necessary to maintain steady motion may be called the **effort**.

Now if the weight in the other scale-pan is increased, it will be found that the effort must be increased still more; the friction, measured as before, will increase with the load. Plot a curve, taking the loads as abscissæ, and the frictions as ordinates. Can you determine from your curve the value of the pulley's friction when the load is *nil* ?

In this simple pulley arrangement no mechanical advantage is gained. The descending weight moves at the same pace as the other. The **velocity ratio** is unity.

§ **117. Moveable pulleys: velocity ratio.** When a **runner** is used (Fig. 172), i.e. a moveable pulley, the string is fixed at one end, and the runner has the weight attached to it. The effort may be measured by a spring-balance, as in Fig. 172, or by a weight passing over a fixed pulley, as in Fig. 173. In the latter case the friction of two pulleys will have to be overcome. Whichever arrangement is adopted, a series of observations with different weights in the scale-pan should be taken, and the values of the effort and the weight lifted, including the runner, should

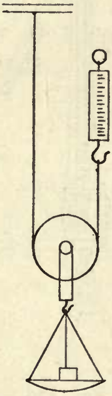


Fig. 172.

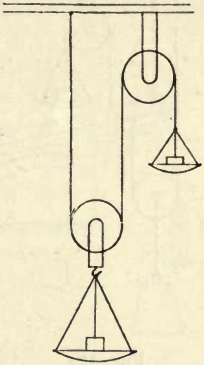


Fig. 173.

be plotted in a curve. Measure also the distances moved through by the end of the string and by the weight attached to the runner. The distances are in the same ratio as the speeds, and so we obtain the **velocity ratio.** If there were no friction this ratio would be the **mechanical advantage.**

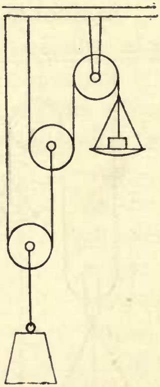


Fig. 174.

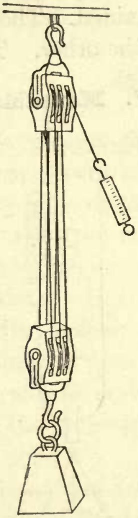


Fig. 175.

It is obvious that, if the two parts of the string on either side of the runner are parallel, the string must be shortened two inches for every inch that the runner is pulled upwards. So the velocity ratio **by calculation** is 2 : 1.

The **effort** moves its end of the string through double the distance travelled by the weight or resistance overcome. Suppose the weight to be 10 lb., and to be pulled up through 1 ft., this means 10 foot-pounds of work. Then the effort will have moved its end of the string through 2 ft., and if the effort has been 6 lb., it has done $6 \times 2 = 12$ foot-pounds of work. Therefore 12 foot-pounds put into the machine will have resulted in 10 foot-pounds given out by the machine, of which the efficiency is

$$\tfrac{10}{12} = \cdot 8\dot{3} = 83\cdot 3 \text{ per cent.}$$

(i) Find the efficiency of your runner with different loads.

(ii) Calculate the velocity ratios of the runner and tackle (Fig. 174), and the purchase (Fig. 175), and find the efficiency of some such arrangements practically with varying loads.

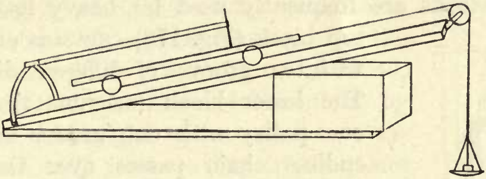


FIG. 176

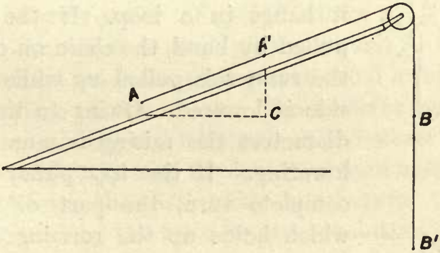


FIG. 177.

(iii) Calculate the velocity ratio of the inclined plane when the inclination of the plane is 30° ; also when the inclination is θ. The efficiency of the apparatus shown in Fig. 176 can be easily measured by adjusting the weight hanging over the pulley until the trolley moves up the plane with uniform speed when started by hand. The work given out is the weight of the trolley multiplied by the vertical height through which it is raised. Thus, if the weight descends from B to B', and the trolley is pulled up from A to A', it is obvious that $BB' = AA'$ (Fig. 177). The work done on the trolley is its weight multiplied by the vertical distance $A'C$ (see § 98).

§ **118. Differential pulley.** Weston's differential pulley blocks are frequently used for heavy loads. The top block (Fig. 178) contains one pulley with two grooves of different diameters. The lower block (or runner) contains one pulley with one groove only. An endless chain passes over the larger groove (the near one in the figure), then round the runner, then over the smaller groove, from which to where we started it hangs in a loop. If the chain be pulled by hand, the chain on one side of the runner is pulled up while the other side is lowered. Owing to the unequal diameters the raising is more than the lowering. If the top pulley makes a complete turn, the part of the chain which holds up the running block will have been shortened by an amount equal to the difference of the circumferences of the two grooves. To do this the hand will have pulled its chain

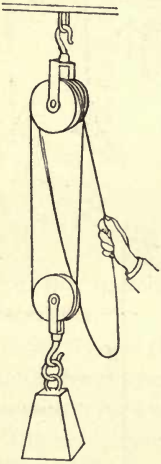


Fig. 178.

down through a distance equal to the circumference of the larger groove.

(i) Show that the velocity ratio is $\frac{2R}{R-r}$ (Fig. 179), and find by experiment the efficiency of a Weston's pulley.

If the efficiency is less than fifty per cent., the friction of the pulleys themselves will prevent the weight from running down when no force is being applied to pull it up. This running down is called **overhauling**, and it is convenient sometimes to have machines which do not overhaul. If the efficiency is fifty per cent., half of the work done is lost in overcoming friction, the **useful work** being equal to the work against friction. As the friction increases in proportion to the load, and almost independently of the effort, by increasing the weight with a view to overhauling we only increase the friction, and the work we try to do in the reverse direction is entirely lost.

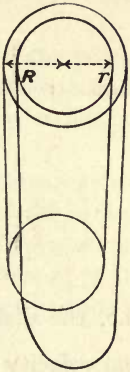


Fig. 179.

§ 119. Screw-jack. Fig. 180 shows a screw-jack, in which a cord passing round a grooved wheel takes the place of the usual lever or *tommy bar* for winding up the screw. The velocity ratio can be measured by making a chalk-mark on the wheel opposite a fixed mark and measuring the length of cord which must be unwound to bring the chalk-mark round again. This is the distance moved by the force in one revolution. The distance moved upwards can be measured directly from the stand by finding the height of a mark on the rim above the stand before and after one complete revolution. The efficiency can be measured as before by finding what weight E in the scale-pan will just lift a known weight W on the jack. Then if the velocity ratio is $\frac{V}{v}$ the efficiency is $\frac{Wv}{EV}$, i.e. $\frac{\text{useful work}}{\text{total work done}}$.

If the radius of the wheel is R, and the pitch of the screw

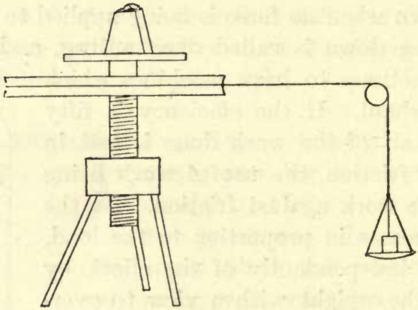


Fig. 180.

(i.e. the distance from thread to thread) is p, show that the velocity ratio must be $\frac{2\pi R}{p}$.

§ **120. Mechanical advantage.** The mechanical advantage of a machine is the ratio of the 'weight' to the 'effort.' For instance, if in a screw-jack a force of 2 lb. lifts a weight of 56 lb., the mechanical advantage is $\frac{56}{2} = 28$. If there were no friction in the machine, this ratio would be the same as the velocity ratio, and the work given out by the machine would be the same as the work put into it. In other words, the efficiency would be unity, or one hundred per cent. If, however, the efficiency of the machine is E, then the work given out $= E \times$ work put in.

Therefore, if $v_1 =$ velocity of effort P,

$v_2 =$ velocity of weight W.

Then $Wv_2 = EPv_1$,

$$\therefore \quad \frac{W}{P} = E\frac{v_1}{v_2}.$$

$\therefore$ mechanical advantage $= E \times$ velocity ratio.

In examination questions the machines are frequently assumed to be frictionless. See § 121 (ix).

§ 121. Exercises:—

(i) Find the relation between the velocities of the power and the weight in the case of an ordinary screw-jack in which the screw is of half-inch pitch and the radius at which the power is applied 2 ft.

The pitch of a screw means the distance between two consecutive threads measured along the axis of the screw, in other words the distance the screw moves forward at each turn. Here the power describes a circle of 2 ft. radius for every turn of the screw. The velocity ratio is therefore $\dfrac{3{\cdot}14 \times 4 \times 12}{0{\cdot}5}$.

(ii) Find the ratio of the power to the weight in the wheel and axle.

If two masses, each of weight w, are fastened to points on the wheel at an angular distance from each other of 120°, find the position of the wheel when the greatest possible weight is supported on the axle, and find what is this greatest weight.

(iii) An ordinary winch is used to lift water out of a well. It is found that a load of 112 lb. hanging on the rope can just be balanced by a force of 30 lb. acting at the handle in a direction tangential to its circular path. The length of the handle is 20 ins., and the diameter of the barrel of the winch is 8 ins. Calculate the efficiency of the winch (Fig. 181).

Work done by power at each revolution $= 30 \times \pi \times 3\frac{1}{3}$ foot-pounds.

Work given out in lifting load $= 112 \times \pi \times \frac{2}{3}$ foot-pounds.

$$\text{Hence efficiency} = \frac{112 \times \frac{2}{3}}{30 \times \frac{10}{3}} = \frac{224}{300} = 74{\cdot}7 \text{ per cent.}$$

(iv) In a system of four pulleys consisting of a fixed block which contains two of them, and a moveable block containing the others, the same rope passes continuously round all the pulleys. From the moveable block is suspended a basket containing a man, the weight of man and basket being *W*. If another man standing on the ground pulls the free end of the rope, what force must he exert to raise the man and the basket?

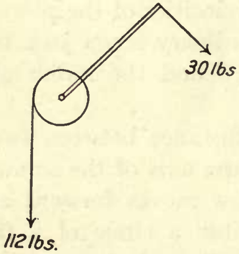


FIG. 181.

If the free end of the rope is pulled by the man in the basket, what force must he exert for the same purpose? (*Army.*)

(v) Find the relation between the power and the weight with the arrangement of pulleys shown in the diagram (Fig. 182), when the weight of the blocks is neglected.

If the weight of the block *A* is 5 lb. and that of *B* 4 lb., what force at *P* will be required to balance a weight of 240 lb.? (*Army.*)

(vi) A system of pulleys is arranged as in Fig. 183, and it is found that it takes a force *P* of 27 lb. to raise a load *W* of 10 lb., it takes a force of 78 lb. to raise a load of 80 lb., and a force of 125 lb. to raise a load of 160 lb. Determine the efficiency of the system in each case. (*Navy.*)

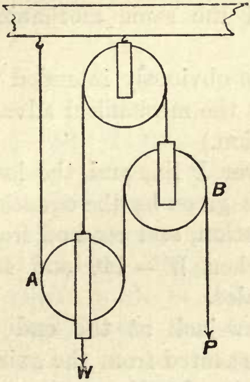


Fig. 182.

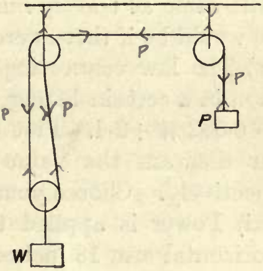


Fig. 183.

(vii) An ordinary block and tackle (the system of pulleys in which there is one continuous rope) has two pulleys in the lower block and two in the upper block. If the system were frictionless, what force would have to be exerted to lift a load of 400 lb.? If the efficiency of the system is actually 0·4, i.e. if a given force will lift only 0·4 times as much as if the system were frictionless, what force would be required to lift the load of 400 lb.? (*Army.*)

(viii) Draw a diagram showing one fixed and four moveable pulleys, arranged so that a separate rope, attached at one end to the supporting beam, passes round each pulley.

Mark on your diagram the tension in each rope when a weight of 260 lb. is being supported, the weight of each pulley being 4 lb. (*Lond. Matric.*)

(ix) What is meant by the *mechanical advantage* of a machine?

Show how to calculate the mechanical advantage of a differential wheel and axle. (See note, p. 141.)

What practical advantage has such a wheel and axle over a simple wheel and axle of the same mechanical advantage?

(Here ' mechanical advantage ' is obviously intended to be the same as *velocity ratio*; what the mechanical advantage would be if there were no friction.)

(x) The law connecting the power P lb., and the load W lb., in a certain lifting tackle, is given by the equation $P = 0{\cdot}052\, W + 2{\cdot}4$. Plot this equation, and read off from your diagram the value of P when $W = 140$ and 460 respectively. Choose your own scales.

(xi) Power is applied to a screw-jack at the end of a horizontal arm 18 inches long (measured from the axis); the screw has a pitch of 1 inch. If there were no frictional losses, what pull would have to be exerted at the end of the arm in order to lift half a ton with this jack?

What would the pull be if only 35 per cent. of the work done on the arm is available for lifting purposes?

(xii) A certain machine is so constructed that a load of W lb. weight can just be lifted by a force P lb. weight given by the formula, $P = 3{\cdot}7 + 0{\cdot}058\, W$, and it is noticed that P must move through 30 feet in order to raise W through 1 foot. Determine the *efficiency* of the machine (i.e. the ratio of the work done on W to the work supplied at P) when W has the values 10, 100, 1000, 10000 in succession.

CHAPTER XIV

PROJECTILES

§ 122. Motion in a curve under a constant force. In the preceding chapter we have considered motion in a straight line only. When a moving body is acted on by a force not in its line of motion, the change of motion will include change of direction. We have spoken of Galileo's work in this connexion (§ 36), and here it is enough to say that Galileo discovered the principle embodied in the second part of Newton's Second Law (§ 46), that change of motion takes place in the direction in which the force acts. In other words, the original motion and the new motion caused by the force are independent of each other. What we mean by this will be understood better by reference to Fig. 184, which gives the path of a particle projected at an angle of 60° with the horizontal with a velocity of 100 ft. per second. If there were no disturbing forces, the particle would move from *A* to *B*, *B* to *C*, *C* to *D*, *D* to *E*, *E* to *F* in the first, second, third, fourth, and fifth seconds respectively: but as the force of gravity is acting all the time independently of this movement, the real position of the particle at the end of the first second is at B^1, 16 ft. vertically below *B*. For in one second the particle would fall through 16 ft., and it does so independently of the velocity of projection which takes the particle from *A* to *B* in that second. The two displacements may in fact be regarded as vectors (§ 15). At the end of the second second the particle, instead of being at C, is at C', 64 ft. vertically below C, the distance fallen being proportional to the square of the time (p. 48). The other points on the path of the projectile, D', E', F', are obtained in the same way, and then are joined by a curve giving the intermediate points. The distance from A to G where the curve cuts the horizontal line gives what may be called the horizontal range of the projectile.

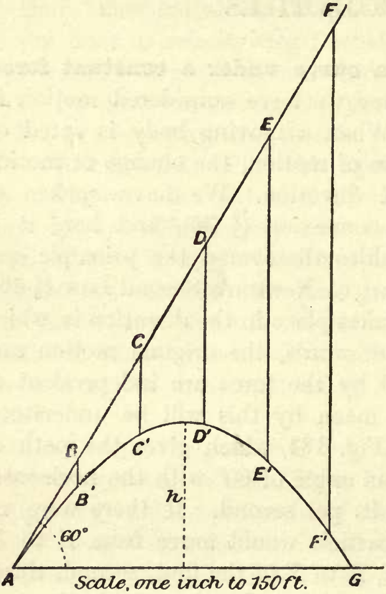


Fig. 184.

(i) Plot on squared paper the path of a projectile, starting with a velocity of 150 ft. per second, at an angle of 50° with the horizontal. Find the horizontal range and the maximum height.

(ii) A man throws a cricket ball with a velocity of 100 ft. per second, at an angle of 45° with the horizontal. Plot the path of the ball and find the length of his throw in yards, the ground being supposed horizontal, and the height of the man's hand above the ground being neglected, as well as the resistance of the air. Find also how the length of the throw would be altered if the angle of projection were (*a*) 60°, (*b*) 30°. Find also the maximum heights in each curve.

(iii) A particle is projected with a velocity of 50 metres per second at an angle of 50° with the horizon up a plane sloping at 10° with the horizon. Plot the path on squared paper and find in metres the range on this plane ($g = 981$ cm.-sec. units).

(iv) Neglecting the resistance of the air, and taking the initial velocity of a stone as 40 feet per second, inclined to the horizon at 40°, draw lines from a given point O to represent in magnitude and direction the velocity of the stone at the end of 1, 2, 3, 4, 5 seconds. From your diagram obtain, by measurement, the inclination of the motion of the stone after $3\frac{1}{2}$ seconds. How does your diagram indicate that the acceleration is constant?

§ 123. Formulæ.

If a particle is projected with a velocity V at an angle a, its position at any subsequent time t seconds after projection can be found by considering V resolved into $V \cos a$ horizontally and $V \sin a$ vertically.

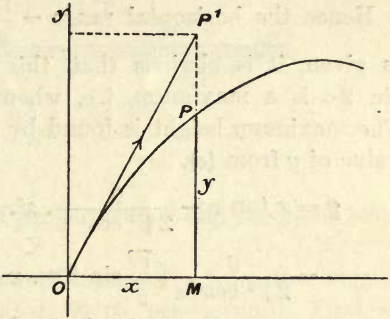


FIG. 185.

It will be seen from Fig. 185 that if P represents the actual position of the particle, and P' what would have been its position but for gravity, then

$$OP' = Vt$$

$$OM = Vt \cos a$$

$$P'M = Vt \sin a$$

$$P'P = \tfrac{1}{2}gt^2.$$

$$\therefore \quad PM = Vt \sin a - \tfrac{1}{2}gt^2.$$

orizontal distance covered = distance = x = $Vt\cos a$ = $\frac{\;}{g}\sin 2a$
at any time at end

The path of the projectile is obtained by giving different values of t to the expressions

(a) $$x = Vt\cos a;$$

(b) $$y = Vt\sin a - \tfrac{1}{2}gt^2.$$

Or by writing, from (a), $t = \dfrac{x}{V\cos a}$, and substituting in (b), we obtain (c) $y = x\tan a - \dfrac{gx^2}{2V^2\cos^2 a}$.

To find the horizontal range, we must put $y = 0$. This gives $x = 0$, and (d) $x = \dfrac{V^2\, 2\sin a\cos a}{g} = \dfrac{V^2}{g}\sin 2a$.

Hence the horizontal range $= \dfrac{V^2}{g}\sin 2a$, and when V is given, it is obvious that this is a maximum when $\sin 2a$ is a maximum, i.e. when $2a = 90°$ and $a = 45°$. The maximum height is found by obtaining the maximum value of y from (c).

$$\begin{aligned} y &= x\tan a - \frac{g}{2V^2\cos^2 a}\,.\,x^2.\\ &= \frac{g}{2V^2\cos^2 a}\left(\frac{V^2}{g}\sin 2a\,.\,x - x^2\right).\\ &= \frac{g}{2V^2\cos^2 a}\left[\left(\frac{V^2\sin 2a}{2g}\right)^2 - \left(\frac{V^2\sin 2a}{2g} - x\right)^2\right].\\ &= \frac{V^2\sin^2 a}{2g} - \frac{g}{2V^2\cos^2 a}\left(\frac{V^2\sin 2a}{2g} - x\right)^2. \end{aligned}$$

y is obviously a maximum when $\dfrac{V^2\sin 2a}{2g} - x = 0$, i.e. when x is half of the horizontal range, and then

(e) $$y = \frac{V^2\sin^2 a}{2g} = \text{maximum height.}$$

(i) Check your results in § 122 (i) and (ii) by substituting for V and a in equations (d) and (e).

To find the range on an inclined plane, the initial velocity may be resolved into two components along and at right angles to the plane, the acceleration of gravity being similarly resolved (Fig. 186).

Let a be the angle of projection, and β the inclination of the plane. It will then be plain that

$$x = Vt\cos(a-\beta) - \tfrac{1}{2}g\sin\beta\, t^2;$$
$$y = Vt\sin(a-\beta) - \tfrac{1}{2}g\cos\beta\, t^2.$$

From this the maximum range may be obtained by putting $y=0$, and finding the corresponding value of t.

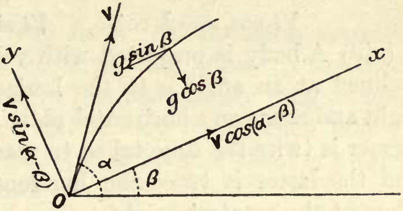


Fig. 186.

(ii) Find the range when $a=50°$, $\beta=10°$, and hence verify your result in § 192 (iii).

(iii) Find the range when the angle of projection is a and the plane has a *downward* slope β.

(iv) A body is projected horizontally from the edge of a cliff with a velocity of 50 ft. per second. Find its positions at the end of the first, the second, and the third seconds, and mark these positions on a diagram. Set off distances on a scale of 50 ft. to the inch.

(v) A stone is projected horizontally from the top of a tower, 256 ft. in height, with the velocity of 128 ft. per second; find when, and where, it will strike the horizontal plane on which the tower is situated, and the direction of its motion at that time.

(vi) Show that the path of a stone thrown into the air in a direction inclined to the horizon would be a parabola if the resistance of the air could be neglected. (You may assume that an equation of the form $Ax^2+Bx=Cy$ represents a parabola.)

(vii) A projectile in vacuo has an initial velocity V. It is required to hit the top of a tower h feet high which stands on the horizontal plane through the point of projection and subtends an angle α at that point. Show how to determine the angle of elevation.

[Let $\theta =$ the angle of elevation.

Distance of tower $= h \cot \alpha$.

If $t =$ time of flight,

$$Vt\cos\beta = h\cot\alpha, \qquad Vt\sin\beta - \tfrac{1}{2}gt^2 = h.]$$

(viii) A body is projected with velocity V in a direction inclined at an angle α to the horizon. Find the time of flight and range on a horizontal plane. Also prove that the former is twice the time taken to reach the greatest height, and the latter is twice the horizontal distance described when at the greatest height.

(ix) A projectile leaves the mouth of a gun with a velocity of 1,500 feet a second. What (to two significant figures) would be its velocity when it had reached a height of 120 feet above the mouth of the gun if the resistance of the air could be neglected? (See § 103.)

CHAPTER XV

CIRCULAR MOTION

§ 124. Newton's First Law. Acceleration towards centre. The tendency of a moving mass to continue moving in the same straight line has already been alluded to in Chap. I, §§ 9–11; and it may be assumed that, if a body is moving in a curve, it is because there is a force continually pulling it out of its straight course. The case of a force constant in direction and magnitude has already been considered under the head of projectiles, and the curve described under such a force has been found to be a parabola. When the curve described is a circle, the direction of motion at any point is always at right angles to the radius of the circle at that point, and if the speed is uniform, the force in this direction must be *nil*; so that the force acting on the moving particle must be towards the centre of the circle.

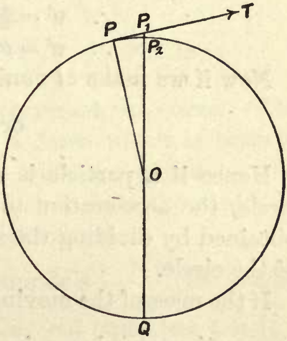


FIG. 187.

Imagine a stone whirled at the end of a string in a horizontal plane, so that the force of gravity does not come in to complicate matters. The radius of the circle OP we will call r; the velocity at P, v.

PT is the tangent at P, and the stone is at this instant moving in the direction PT. In a small interval of time, δt, the stone would move to P_1, where $PP_1 = v\delta t$. But it is brought to P_2 instead of P_1 by the pull of the string, which produces an acceleration which we will call a. Assuming this acceleration to be constant during the small interval of time δt, we have, by the formula of § 19,

$$P_1P_2 = \tfrac{1}{2}a(\delta t)^2.$$

The smaller we make the interval δt, the more exactly is this statement true.

But from the figure it is plain that

$$PP_1^2 = \text{rect. } P_1P_2 \,.\, P_1Q,$$

$$\therefore \quad (v\delta t)^2 = \tfrac{1}{2}a(\delta t)^2(2r + \tfrac{1}{2}a(\delta t)^2),$$

$$\therefore \quad v^2 = \tfrac{1}{2}a(2r + \tfrac{1}{2}a(\delta t)^2),$$

$$\therefore \quad v^2 = ar + \tfrac{1}{4}a^2(\delta t)^2.$$

Now if we make δt vanish, we find that $v^2 = ar$,

$$\text{or} \quad a = \frac{v^2}{r}.$$

Hence if a particle is moving with uniform speed in a circle, the acceleration towards the centre of the circle is obtained by dividing the square of the speed by the radius of the circle.

If the mass of the moving particle be m, the measure of the *force* acting on it is ma, i.e. $\dfrac{mv^2}{r}$. The units must, of course, be properly chosen. Thus if the force is to be expressed in dynes, m must be measured in grammes, v in cm. per second, and r in cm.

(i) A smooth ball of 50 grammes mass moves on a smooth horizontal table at the end of a string 30 cm. long, the other end of which is fastened to a fixed point on the table. If the speed of the ball is 6 metres per second, find the tension of the string in dynes.

(ii) If in the previous exercise the pull on the string had been equal to the weight of the ball, what must the speed have been?

(iii) An engine of 10 tons mass runs round a curve of half-mile radius at a speed of 20 miles an hour. Find in lb. weight the outward pressure on the rails, or the inward pressure of the rails on the engine.

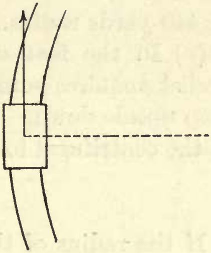


FIG. 188.

Here again we have an example of circular motion, and of the so-called 'centrifugal force,' a term which may be misleading unless it is clearly understood to be the force with which the engine presses outwards on the rails in its attempt to obey Newton's First Law and go on in a straight line instead of coming round the curve. The rails pull the engine round with a force which is *towards* the centre, and consequently the rails are pulled by the engine with an opposite force which is *from* the centre of the curve, and so may be called centrifugal.

In this case the force $= \frac{mv^2}{r}$ poundals $= \frac{Wv^2}{rg}$ lb. weight if m, v, r, g, W are in foot-pound-second units (see § 104).

$$m = 10 \text{ tons} = 10 \times 2240 \text{ lb.}$$

$$v = 20 \text{ miles an hour} = \frac{20 \times 1760 \times 3}{60 \times 60} \text{ ft. per sec.}$$

$$= \tfrac{88}{3} \text{ ft. per sec.}$$

$$r = \tfrac{1}{2} \text{ mile} = 880 \times 3 \text{ ft.}$$

$$g = 32.$$

$$\therefore \text{ force in lb. weight} = 10 \times 2240 \times \frac{88^2}{3^2} \times \frac{1}{880 \times 3 \times 32}.$$

$$= \frac{2240 \times 11}{27 \times 4} = \frac{11 \times 2240}{108}.$$

$\therefore$ outward pressure on rails $= \frac{11}{108}$ of the weight of a ton.

(iv) If the outward pressure of the wheels of an engine on the rails must not exceed one per cent. of the weight of the engine, find the maximum speed allowable round a curve of 440 yards radius.

(v) In the feat of 'looping the loop,' the performing cyclist acquires sufficient velocity to carry him round the loop upside down. Here we have gravity directly opposed to the centrifugal force, and therefore at the top of the loop

$$\frac{mv^2}{r} \not< mg.$$

If the radius of the upper portion of the loop is 20 ft., what must the speed of the cyclist be at this point?

§ 125. Conical pendulum: simple pendulum.

If a heavy ball B is hung from the end of a string fastened at A (Fig. 189), it can be made to swing in a circle under the action of gravity. The movement of the ball appears to an observer, whose eye is in the plane of the circle, to be backwards and forwards in a straight line, and the time occupied in a complete revolution is apparently the same as that of a backwards and forwards swing when the ball is swinging like a pendulum between the same extreme points. Observe that while the ball describes a circle, the string describes a cone. This is, therefore, known as the *conical pendulum.*

Now let us consider the forces acting on the ball. They are the tension of the string T, and the weight of the ball mg.

There is no vertical acceleration, as the ball moves in a horizontal plane. Therefore the vertical force on the ball is *nil*, i.e. $T \cos \theta - mg = 0$ (Fig. 190).

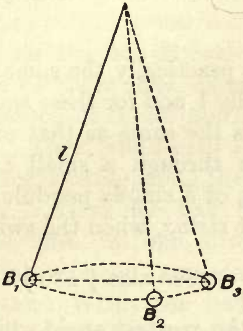


FIG. 189.

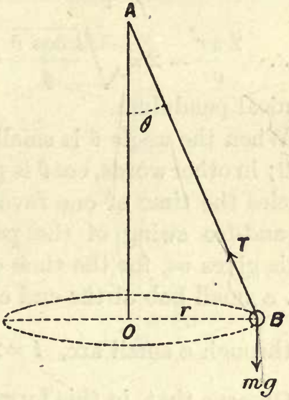


FIG. 190.

The acceleration towards the centre of the circle O is $\frac{v^2}{r}$, therefore the force pulling the ball towards the centre is $\frac{mv^2}{r}$, and this can only be caused by the pull of the string resolved horizontally. Hence $T \sin \theta = \frac{mv^2}{r}$.

This gives us, in conjunction with the other equation,

$$mg \tan \theta = \frac{mv^2}{r},$$

$$\therefore \quad v^2 = rg \tan \theta$$

$$= rg \frac{r}{AO}$$

$$= \frac{r^2 g}{l \cos \theta}.$$

Now the circumference of the circle is $2\pi r$, and therefore the time taken to describe a complete circle is

$$\frac{2\pi r}{v}; \quad \text{but} \quad \frac{r^2}{v^2} = \frac{l \cos \theta}{g}.$$

$$\therefore \quad \frac{2\pi r}{v} = 2\pi\sqrt{\frac{l \cos \theta}{g}} = \text{time of one revolution of a}$$

conical pendulum.

When the angle θ is small, AO is practically the same as AB; in other words, $\cos \theta$ is practically 1, and for these small circles the time of one revolution is the same as that of a to-and-fro swing of the pendulum through a small arc. This gives us, for the time of swing of a simple pendulum, i.e. a small bob at the end of a light string, when the swing is through a small arc, $t = 2\pi\sqrt{\frac{l}{g}}$. (See also § 130.)

Observe that, in this formula, θ, the greatest angle which the string makes with the vertical, does not occur: in other words, so long as θ is small enough for $\sqrt{\cos \theta}$ to be taken as unity, the time of swing is independent of the amplitude of the swing. (The amplitude is half the distance between the extreme points.)

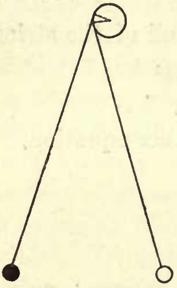


FIG. 191.

(i) A ball at the end of a string 50 cm. long swings as a conical pendulum. Find the time of a complete revolution when the semi-vertical angle of the cone θ (see Fig. 190) is (*a*) 10°, (*b*) 5°. ($g = 981$. Give results to three places of decimals of a second.)

(ii) Hang up a heavy ball by a string as light and as long as possible (at least a metre). The point of suspension should be as definite as possible. Fig. 191 shows a bad suspension, as the string winds and unwinds itself about the nail round which it is wrapped. Fig. 192 shows a better arrangement. Measure carefully the distance from *A* to the centre of the ball *B*. Then mark on the floor or the table, immediately below *B*, two chalk lines at right angles, or pencil lines on a sheet of paper pinned down.

Pull the ball *B* back along one of these lines through a distance of 5 cm. and let it go. Then stand with a stop-watch looking along the other line. When the ball crosses this line in one direction, count 'nought' and start the watch. When the ball next crosses the line in the same direction, count 'one,' and so on, until you have counted 'twenty,' at which instant you must stop the watch, which should record the time of twenty complete (to-and-fro) swings. Now repeat the observations with an initial displacement of 10 cm. instead of 5. Enter your results in tabular form, thus :—

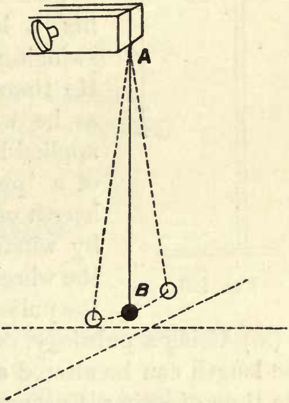


FIG. 192.

Length of pendulum = cm.		
Displacement.	Time of twenty swings.	Time of one swing.

The experiment may be repeated with a ball of different weight, but with the same length of string. In this way you may verify the fact that the pendulum is *isochronous*, i.e. that the swings take the same time whether they are long or short, provided that they are small in comparison with the length of the string.

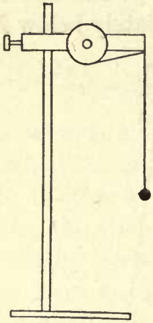


FIG. 193.

Galileo discovered this fact by watching a lamp in the cathedral at Pisa swinging by its long chain from the roof. He timed the swings by his pulse; and, as he was then studying medicine, he applied his discovery to the construction of a 'pulsilogy' (Fig. 193), in which the length of the pendulum can be adjusted by winding or unwinding the string on the wheel, until its swings keep time with the pulse of a patient.

(iii) Using a pulsilogy, or any simple pendulum in which the length can be altered and measured easily, verify that the time of swing is proportional to the square root of the length. Then from the formula $t = 2\pi\sqrt{\frac{l}{g}}$, calculate the value of g.

This method of determining the value of g is much more accurate than that of direct observations of the motion of falling bodies; and when allowance is made for the mass of the string, the size of the ball, the resistance of the air, and the length of the arc, it is capable of giving great accuracy.

(iv) Find the length in inches of a simple pendulum beating seconds, i.e. when each half-swing takes 1 second, taking $g = 32{\cdot}18$.

(v) The pendulum of a clock consists of a heavy bob at the end of an iron rod of length l. The pendulum beats seconds; but when the temperature rises, the iron expands and the clock goes slow. If the temperature rises 20° C., how much will the clock lose in 24 hours? (Coefficient of expansion of iron $=\cdot0000117$.)

Time of half-swing $= \pi\sqrt{\dfrac{l}{g}} = 1$ second.

Number of swings per diem

$$= \frac{24\times 60\times 60}{\pi}\sqrt{\frac{g}{l}} = 24\times 60\times 60.$$

When the temperature rises 20°, l becomes

$$l + \cdot0000117\times 20\,l = l\,(1+\cdot000234).$$

Number of swings lost

$$= \frac{24\times 60\times 60\sqrt{g}}{\pi\sqrt{l}}\left(1-\frac{1}{\sqrt{1\cdot000234}}\right).$$

$$= \frac{24\times 60\times 60}{\pi}\sqrt{\frac{g}{l}}\,(\cdot000117).$$

$$= 24\times 60\times 60\times\cdot000117 = 10 \text{ seconds approximately.}$$

§ **126. 'Centrifugal force.'** When a truck runs round a curve, the flanges on the wheels may to some extent prevent the truck running off the rails; but the pressure P of the rail on the flange (Fig. 194) does not act through G, the centre of gravity of the truck. Hence there is a *couple* which may upset the truck. If the truck were on the point of turning over, the whole of the weight would be on the outside rail, so that the couple

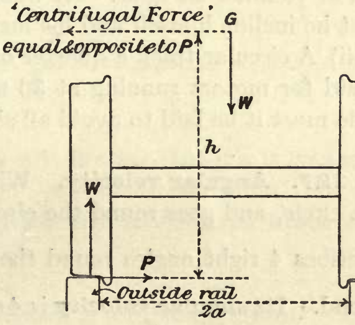


FIG. 194.

WW (see Fig. 194) is just sufficient to balance the other couple. If $2a$ = distance between the rails, and h = height of centre of gravity of the truck, m its mass, v its velocity, and r the radius of the curve round which it is running, we have

$$Ph = Wa \text{ (since the couples have equal moments).}$$

But
$$P = \frac{mv^2}{r}, \quad W = mg,$$
$$\therefore \quad \frac{v^2 h}{r} = ga.$$

The same result would be obtained by taking moments about the outer rail, the ‘ centrifugal force ’ being balanced by the weight.

(i) A car with wheels 4 ft. 6 ins. apart, and with its centre of gravity 2 ft. 6 ins. from the ground, goes round a corner at 20 miles an hour. Find the radius of the sharpest curve that it can take without upsetting.

(ii) A cyclist goes round a curve of 20 ft. radius at the rate of 10 miles an hour. At what angle with the vertical must he incline himself and his machine?

(iii) A circular track a quarter of a mile in circumference is laid for motors running at 30 miles an hour. At what angle must it be laid to avoid all side-slips?

§ **127. Angular velocity.** When a particle is moving in a circle, and goes round the circle n times per second, it describes 4 right angles round the centre in $\frac{1}{n}$th part of a second. Its **angular velocity** is $4n$ right angles per second, or $360n°$ per second, or $2\pi n$ radians per second. Circular measure is usually employed in measuring angular velocity. Thus if an angular velocity is expressed by the letter ω, this would mean that ω radians per second are described round a point; the time of a complete revolution of 4 right angles would be $\frac{2\pi}{\omega}$ seconds.

(i) What is the angular velocity of (*a*) the minute-hand, (*b*) the hour-hand of a watch? Are the results the same for a watch with hands 1 in. long, and a clock with hands 24 ft. long (like Big Ben)?

Fig. 195 shows the angle *AOB* as ω, the angle described per second,

$$\frac{\text{arc } AB}{OA} = \omega; \quad \frac{\text{arc } A'B'}{OA'} = \omega.$$

If $OA = r$, $OA' = r'$, we have

$$\text{arc } AB = r\omega; \quad \text{arc } A'B' = r'\omega.$$

And these arcs give the actual distances described in 1 second by points distant r and r' from O.

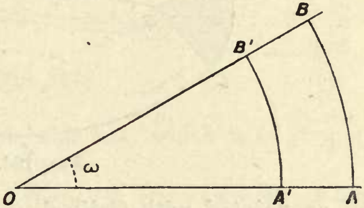


Fig. 195.

What are the velocities of the ends of the minute-hands of the clock and watch previously mentioned?

In the case of a particle moving in a circle of radius r with velocity v, it is clear that if ω is its angular velocity, then $v = r\omega$.

Hence the expression for the acceleration towards the centre is $\frac{r^2\omega^2}{r} = r\omega^2$.

This form of the expression is sometimes useful.

(ii) If the earth rotates about its axis in 23 hours 56 minutes, and if its equatorial diameter is 7,925 miles, determine the linear velocity in feet per second (to the nearest hundred) of a point at the equator. What is the angular velocity of this point in radians per hour (to the nearest hundredth)? (*Army.*)

(iii) The moon revolves round the earth (considered as a fixed point) in about 27 days 8 hours. What is the angular velocity? What is the acceleration of the moon towards the earth if the radius of the moon's orbit be 240,000 miles? How does this compare with the acceleration of gravity at the earth's surface?

§ **128. Universal gravitation.** Isaac Newton, when a young man at Trinity College, Cambridge, had made himself acquainted with Galileo's work, and had investigated the question of circular motion. He had gone further than this. Kepler had discovered that the irregular movements of the planets in the heavens could be explained by supposing the planets to be moving in ellipses with the sun at one of the foci of the ellipse, and with angular velocities such that the line joining a planet to the sun swept out equal areas in equal times (Fig. 196).

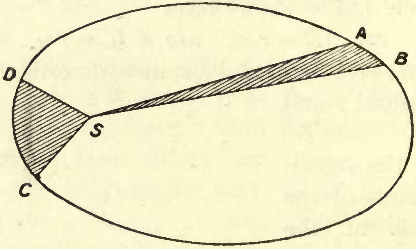


FIG. 196.

Thus when a planet is far away from the sun it moves slower than when near the sun, so that if *AB*, *CD* are arcs described by the planet in equal times, the shaded area *ASB* is equal to the shaded area *CSD*.

These discoveries, and another connecting the periods of revolution of the different planets, were stated by Kepler in his three laws.

Newton investigated this motion of the planets mathematically, and found that it would be a necessary consequence if there existed between the sun and the planets a force like the pull of a rope, but variable in strength, and varying so as to decrease when the distance increased according to the law of 'inverse squares,' e.g. if the distance is doubled, the force is $\frac{1}{4}$ of its former value, if the distance is $\frac{1}{3}$ of its former value, the force is multiplied by 9.

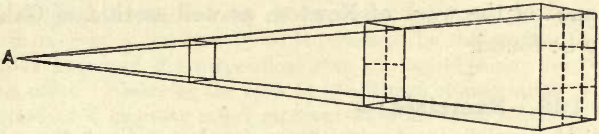


Fig. 197.

Fig. 197 helps to explain this law, which is of frequent occurrence in nature.

But what evidence was there for the existence of such a force? The story goes that Newton, exiled from Cambridge on account of the plague, was thinking about this subject as he sat in the orchard at Woolsthorpe when the apple fell. Here was the same mysterious attractive force. Was it the same? Did it obey the law of inverse squares? If so, the attraction of the earth on the moon ought to be such as to produce an acceleration considerably less than that of gravity. How much less? The radius of the earth is about 4,000 miles, the distance of the moon from the earth about 240,000 miles. What fraction of g ought the acceleration of the moon towards the earth to be?

Newton, on working this out for the first time, relied on an incorrect measurement of the earth, and he found the two accelerations were not in the right proportion according to his theory. The discrepancy was as much as 15 per cent., and he did not work at his theory any further for the time. Some years afterwards he heard of a fresh measurement of an arc of the earth's circumference made in France, which gave a value of about 4,000 miles for the radius of the earth. Newton at once tried his figures again, with the wonderful discovery of universal gravitation in his grasp, and for the next two years he worked at the subject with results that are embodied in his *Principia.* The reader should consult Lodge's *Pioneers of Science,* where a most interesting account of the work of Newton, as well as that of Galileo, may be found.

§ 129. **Exercises :—**

(i) A particle is moving in a circular path, 5 ft. radius, at a speed of 30 ft. per second. If the mass of the particle is 10 lb., what force must act to constrain the motion? Make a sketch, showing clearly how the force must act supposing the particle is constrained in its path by a string attached to it and to the centre of the path. What would be the motion of the particle if the string were cut?

(ii) If the moon revolves about the earth in a circle whose radius is 240,000 miles, performing one revolution in 30 days, find its velocity and acceleration in foot-second units.

(iii) A particle moves with uniform speed in a circle of radius r feet, making n revolutions per second. Find an expression for the acceleration towards the centre.

(iv) The moon has a mass of $1{\cdot}7 \times 10^{24}$ lb., and describes a circular orbit about the earth of radius $1{\cdot}26 \times 10^{9}$ ft. in 27 days 8 hours. Calculate the moon's acceleration and the force that produces it. State your units.

(v) A heavy particle is attached to one end of a string 3 ft. in length, the other is fastened to a fixed point; and the particle is held so that the string is inclined at the angle of 60° to the vertical. Find the velocity with which it must be projected horizontally so as to describe, uniformly, a horizontal circle. (*Army.*)

Moment of Momentum.—When a body, such as a bar, or a wheel, is revolving about a fixed point, it is obvious that the parts furthest away from that point are moving fastest. The effect of a blow will be to change the velocities of every point of the body, and therefore to create suddenly a set of internal actions and reactions (§ 54). If the body keeps its shape, in other words, if the body is *rigid*, these internal stresses will not affect its motion; but the whole body will be moving with a different angular velocity on account of the blow. The turning effect or *moment* of the blow (§ 75) is measured by the product of the impulsive force and the perpendicular on its line of action from the turning-point. Measuring the blow by the change of momentum (§ 55), we introduce a quantity called moment of momentum, which is the product of the mass, the velocity, and the afore-mentioned perpendicular; or if we please we can resolve the momentum into two components, one through the turning-point, and the other at right angles to this line, and obviously it is only the latter component which has any moment. If the distance from the turning-point be r, and the angular velocity ω (§ 127), this resolved momentum is $mr\omega$, and the moment of momentum is $mr^2\omega$. But r is different for every part of the rigid body, and the total moment of momentum must be $\Sigma mr^2\omega$ (see § 107), in other words, it is the moment of inertia multiplied by the angular velocity. By Newton's Third Law we should arrive at the following conclusion:—When a collision occurs between two bodies, one of which has to rotate about a fixed point, the total moment of momentum is unchanged. Mr. Fletcher's apparatus for verifying this fact, and showing the constancy of the moment of inertia, is described in the *School World* (May, 1904). The trolley of § 42 runs against a lath mounted to rotate in a horizontal plane. The lath is a long one, and carries a crosspiece at one end, on which the angular velocity can be recorded by a vibrating paint-brush, as is done in the case of the trolley.

This important principle is known as 'The Conservation of Moment of Momentum.'

CHAPTER XVI

SIMPLE HARMONIC MOTION

§ **130. Simple harmonic motion.** Consider a particle P of mass m, moving in a circle, centre O, with uniform speed. The force pulling it towards the centre of the circle is $\frac{mv^2}{r}$, or $mr\omega^2$, if ω is the angular velocity. This force may be resolved into two components, one parallel to the line Ox, the other parallel to Oy (Fig. 198).

The force parallel to Ox is $mr\omega^2 \cos \theta$.

But
$$\cos \theta = \frac{OM}{OP} = \frac{x}{r},$$

$\therefore$ the force parallel to Ox is $mx\omega^2$. Similarly the force parallel to Oy is $my\omega^2$.

Now the circular movement of the point P may be regarded as made up of two independent movements in directions PN, PM. The movement in the direction PN is the same as the movement of the point M which is always directly underneath P. And this movement is due to the force $mx\omega^2$. It follows then that a particle of mass m at a distance x from O acted on by a force $mx\omega^2$ towards O will move backwards and forwards as the point M does. The motion is precisely the same as the apparent movement of the conical pendulum (Fig. 189) to an observer at some distance off whose eye is in the plane in which the bob of the pendulum is moving.

Such a movement is called a simple harmonic motion (sometimes S.H.M.), and it very closely resembles the swinging of an ordinary pendulum (§ 125) or the vibration of a steel spring or tuning-fork.

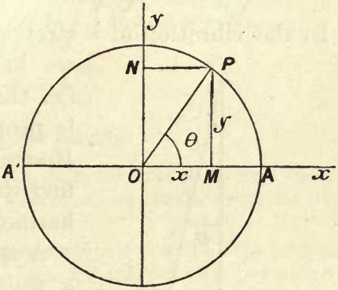


FIG. 198.

Note that (*a*) the force causing the motion is towards the centre and proportional to the distance from the centre.

(*b*) The motion is periodic, i.e. the particle keeps returning to its former position, and describing the same path over again.

(*c*) The period, i.e. the time taken to describe the complete path (' the interval between the passing of a given point, and the next passing of that point in the same direction '), is the same as the time taken by the point P to complete the circle. This time is $\frac{2\pi r}{v}$, or $\frac{2\pi}{\omega}$.

Now the force is $mx\omega^2$, and the acceleration is $x\omega^2$, where ω^2 is the constant of variation. Suppose that the acceleration towards the centre is μ times the distance, i. e. μx. Then $\mu = \omega^2$, and the period is $\frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\mu}}$.

As an example, we may take the simple pendulum already described (§ 125), and consider the force tending to bring the bob back to the vertical position. It is

$$mg \sin\theta = \frac{mgx}{l}.$$

Hence the acceleration, for angles small enough for us to regard the arc as a straight line, is proportional to the displacement x, and is $\frac{gx}{l}\cdot$ Here $\mu = \frac{g}{l}$, and the time of swing $= \frac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\frac{l}{g}}$.

In the vibration of a stretched string or a spiral spring, we know that the displacement, i.e. the extension or compression, is proportional to the stress, by Hooke's law (§ 140). Hence we find that the vibration is a simple harmonic motion.

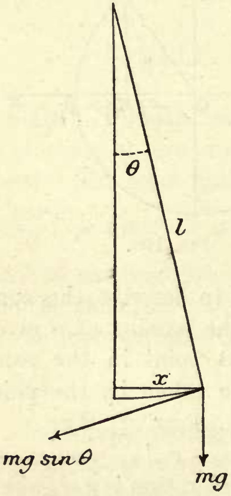


FIG. 199.

A spiral spring is stretched by a weight of 100 grammes. If 5 grammes were added, the spring would stretch an additional distance of 5 mm. Find the time of an oscillation when the spring vibrates with the 100 grammes.

When the spring is displaced through 5 mm., there is a force equal to the weight of 5 grammes tending to bring it back. As this force is proportional to the strain or displacement we have the acceleration also proportional. The acceleration is $\frac{5g}{100}$ and the displacement is 0·5 cm.

$$\frac{5g}{100} = \mu \times 0{\cdot}5,$$

$$\therefore \quad \mu = \frac{g}{10} = 98 \text{ (C. G. S. units).}$$

$$\text{Time of oscillation} = \frac{2\pi}{\sqrt{98}} = 0{\cdot}634 \text{ second.}$$

§ **131.** The angular velocity of a point moving in a circle of radius r being ω, it is plain that the actual velocity is $r\omega$. Resolving this along Ox (Fig. 198) we see that the velocity of the point M is $-r\omega \sin\theta$. This may be obtained in another way $x = OM = r\cos\theta$.

$$x + \delta x = r\cos(\theta + \delta\theta) = r(\cos\theta - \sin\theta\delta\theta),$$

since when $\delta\theta$ is very small, $\cos\delta\theta = 1$ and $\sin\delta\theta = \delta\theta$.

This gives $$\delta x = -r\sin\theta\delta\theta.$$

$$\therefore \quad \frac{\delta x}{\delta t} = -r\sin\theta\frac{\delta\theta}{\delta t}.$$

But $$\frac{\delta\theta}{\delta t} = \omega, \text{ the angular velocity.}$$

Again, the acceleration of the point M is the rate of change of its velocity. Taking this velocity as $-r\omega\sin\theta$, we obtain as its rate of change $-r\omega^2\cos\theta = -\omega^2 x$.

The negative sign means that the acceleration is in a direction tending to decrease x, just as the negative sign for the velocity $(-r\omega\sin\theta)$ means that x diminishes as θ increases as long as $\sin\theta$ is positive, i.e. during the passage from A to A' (Fig. 198).

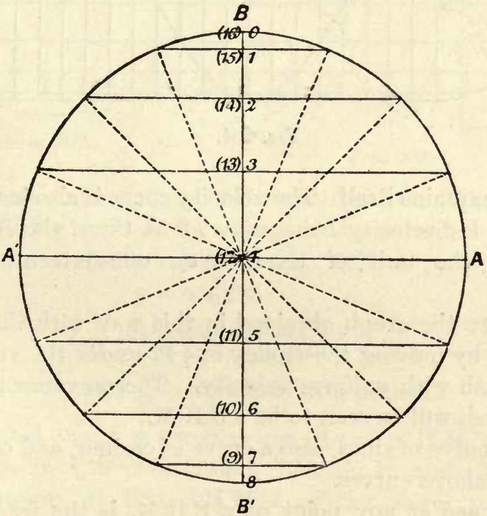


Fig. 200.

§ 132. Space and velocity graphs of a S.H.M. It is obvious from Fig. 200 that a S.H.M. of amplitude equal to *OB* can be divided into equal intervals of time by dividing up the circle *ABA′B′* into equal arcs. This has been done in the figure, and if we suppose the period divided into 16 equal intervals, starting from the extreme position *B*, the particle, which always keeps level with an imaginary particle moving uniformly in the circle, will move in the first of these intervals from 1, in the second from 1 to 2, &c.

It now becomes very easy to draw the space-graph, and

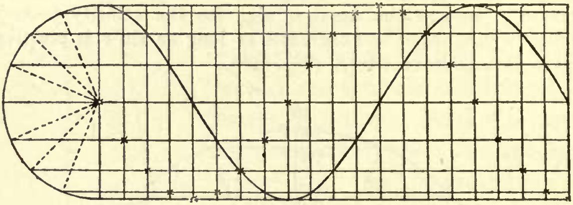


Fig. 201.

Fig. 201 explains itself. The velocity curve is also indicated, the scale for velocity being about 2·54 times the scale for distance, the unit of time being one-sixteenth of a period.

Compare the graph obtained in this way with the curve obtained by moving the trolley of § 42 under the vibrating paint-brush with uniform velocity. The movement of the paint-brush will be seen to be a S.H.M.

Plot a curve of sines, also a curve of cosines, and compare with the above curves.

The phase at any point of a S.H.M. is the fraction of a period that has elapsed since the particle last passed through its point of maximum displacement. Thus the phase corresponding to P (Fig. 198) is $\frac{\theta}{2\pi}$.

§ 133. Two simple harmonic motions at right angles. It is obvious that circular motion is a combination of two simple harmonic motions of the same period and amplitude, but differing in phase by a quarter of a period.

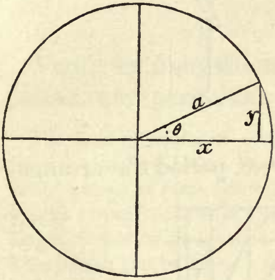


Fig. 202.

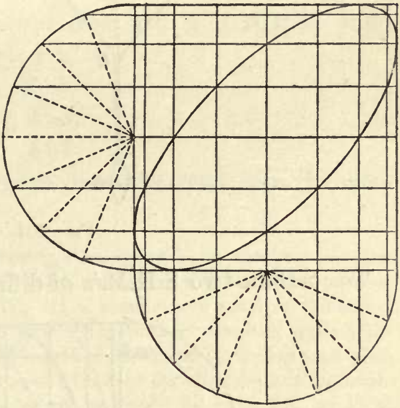


Fig. 203.

For

$$x = a \cos \theta \text{ (Fig. 202)},$$
$$y = a \sin \theta,$$
$$= a \cos\left(\theta - \frac{\pi}{2}\right),$$

and $\frac{\pi}{2}$ is a quarter of a period.

Fig. 203 shows a combination of two S.H.M.'s differing in phase by $\frac{1}{8}$th of a period. Obtain in a similar way a combination of two S.H.M.'s of the same period but (α) differing in phase by $\frac{1}{16}$th period, (β) of the same period.

A pendulum dropping sand may be used to illustrate these combinations, if it is capable of receiving a conical motion, as in § 125.

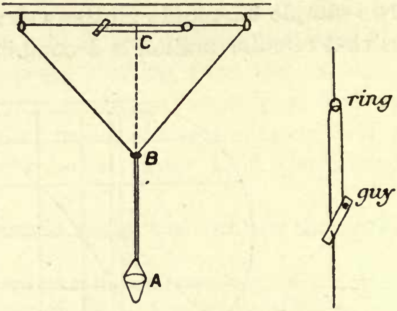


Fig. 204.

To combine two S.H.M.'s of different period the arrangement shown in Fig. 204 is useful. To obtain periods in the ratio 1 : 2, make $AB = \frac{1}{4}\ AC$ (see § 125). The curve obtained by such an arrangement when the two motions

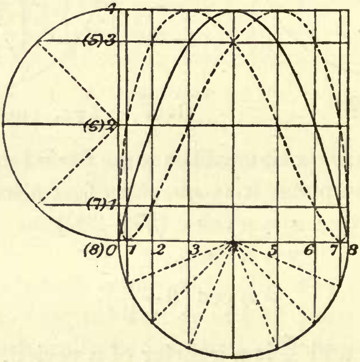


Fig. 205.

start at the same phase is shown in Fig. 205. The dotted curve shows a difference in phase, and if the lengths *AB*, *AC* are not correctly adjusted, the sand curve will go through a series of different shapes, the changes coming more quickly if the inaccuracy of adjustment is increased. This illustrates the principle of Lissajou's method of adjusting tuning-forks.

Draw the curves obtained from two S.H.M.'s at right angles of which the periods are

(*a*)	2 : 3
(*b*)	3 : 4
(*c*)	4 : 5.

Verify by comparison with those obtained with the compound sand pendulum.

The time of swing of a vibrating mass depends on the way in which the mass is distributed, its *moment of inertia*, as well as on the position of its centre of mass (see § 107). If a small heavy mass at the end of a light string vibrates in exact time with another vibrating system, the length of this string is called the length of the *simple equivalent pendulum.* This length can be found very approximately for vibrations of moderate frequency by adjusting the pulsilogy described on page 224, so as to keep exact time with the other vibrating system.

CHAPTER XVII

ELASTICITY. STRESS AND STRAIN

§ **134. Elasticity.** When two bodies collide they do not, as a rule, remain in contact; but there is a rebound, the extent of which depends partly on the relative speed at impact, partly on the nature of the materials of the colliding bodies. The student should repeat Newton's experiments with balls of different materials (see Figs. 45 and 46) and find what ratio exists between the relative velocities of the two spheres before and after impact. Remember that velocities in opposite directions (or senses) must be given opposite signs, and that the relative velocity is the difference of the two velocities, i.e. $v_1 - v_2$, due regard being paid to sign. Thus if two trains are moving with velocities of 60 and 40 miles an hour, their relative velocity is 20 miles an hour if they are going in the same direction, 100 miles an hour if in opposite directions.

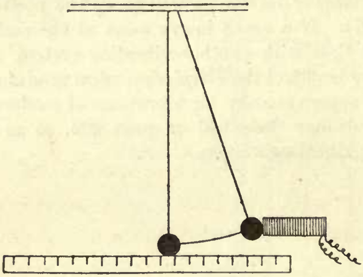


FIG. 206.

In experimenting with steel spheres, it is convenient to have an electro-magnetic release (see Fig. 206), which will hold the ball back in position till a key is pressed down to break the current.

It will be found that the relative velocities before and after impact bear a constant ratio to each other. The velocities may be measured as in using Hicks's ballistic pendulum (§ 52). If glass balls or steel motor-car bearings of equal mass are being used, and if one of these is allowed to impinge on the other at rest, the first will be very nearly stopped dead, and the velocity given to the other will be not much less than the velocity which the first has lost. Find the ratio for steel for different positions of the electromagnet. If balls of putty are used, there will be no rebound, the relative velocity being zero, as the balls will remain in contact.

From a series of experiments such as these, Newton arrived at the conclusion that the relative velocities before and after impact bore a constant ratio for the same substances, the ratio being always less than 1. The relation may be expressed thus, $\frac{v_1 - v_2}{u_1 - u_2} = -e$,

where u_1, u_2 are the velocities of the two spheres before impact, v_1, v_2 their velocities after impact. e is always less than 1, and it is given a negative sign in the above equation because the relative velocity is always reversed in direction. This only means that the spheres approach before and recede after impact, which is sufficiently obvious.

§ 135. Coefficient of restitution. e is called the coefficient of restitution; sometimes, but wrongly, the coefficient of elasticity. If we consider what happens at a collision, we find that the spheres remain in contact for a short time, losing their shape temporarily. That this is the case can be seen by dropping a steel or glass ball on a dusty slab. There will be a circular mark on the dust at the point of contact, the area of this mark increasing with the height from which the ball is dropped. Now this can only mean that the sphere is more or less flattened by the impact, recovering its shape afterwards. Hence some of the kinetic energy of the sphere is converted into potential energy during the first part of the contact. Some of it is lost in heat and sound vibrations ; but the stored potential energy becomes again kinetic energy as the sphere recovers its shape. However there is less kinetic energy after impact than before, owing to the loss already alluded to. If the kinetic energies before and after impact are E_1, E_2, then $E_1 - E_2$ is the lost energy. If e is the kinetic energy at the middle of the impact, $E_1 - e$ is the energy which ought to have been stored up as potential energy. The lost energy is a fraction of this which depends on the materials of the bodies. Let k be this fraction. Then

$$E_1 - E_2 = k(E_1 - e),$$

which gives

$$E_2 - e = (1-k)(E_1 - e).$$

Now

$$E_1 = \tfrac{1}{2} m_1 u_1^2 + \tfrac{1}{2} m_2 u_2^2$$

$$E_2 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$

and, if V be the velocity of the two spheres at the middle of their contact,

$$e = \tfrac{1}{2}(m_1 + m_2) V^2.$$

But

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 = (m_1 + m_2) V \text{ (see § 53)},$$

$$\therefore \quad E_1 - e = \tfrac{1}{2} m_1 u_1^2 + \tfrac{1}{2} m_2 u_2^2 - \tfrac{1}{2} \cdot \frac{(m_1 u_1 + m_2 u_2)^2}{m_1 + m_2}$$

$$= \frac{m_1 m_2 [u_1^2 + u_2^2 - 2 u_1 u_2]}{2(m_1 + m_2)} = \frac{m_1 m_2}{2(m_1 + m_2)} (u_1 - u_2)^2.$$

Also

$$E_2 - e = \frac{m_1 m_2}{2(m_1 + m_2)} (v_1 - v_2)^2,$$

$$\therefore \quad \frac{E_2 - e}{E_1 - e} = \frac{(v_1 - v_2)^2}{(u_1 - u_2)^2},$$

$$\therefore \quad 1 - k = \left(\frac{v_1 - v_2}{u_1 - u_2}\right)^2.$$

This gives

$$e = \sqrt{1-k}.$$

(i) When the coefficient of restitution is $\frac{15}{16}$, what fraction of the kinetic energy is lost at impact?

(ii) If the loss of kinetic energy at impact is 10 per cent., what is the ratio of the relative velocities before and after impact?

§ 136. Impact on a fixed surface.

When one of the bodies is fixed, e.g. when a billiard ball strikes a cushion, we have, of course, $u_2 = v_2 = 0$.

Hence $v_1 = -eu_1$, and the velocity of rebound is $-e$ times the velocity before impact. When the ball strikes the cushion at an angle, the angle of rebound is determined partly by the coefficient of restitution, but it also depends on the way in which the ball is rotating. The cushion stops the spin and so affects the motion of the ball in a direction parallel to itself. Assuming that the motion along the cushion is unaffected, which would be the case if there were no friction between the ball and the cushion, we may resolve the velocity before impact into $v \cos a$ perpendicular to the cushion and $v \sin a$ along the cushion (Fig. 207); $v \sin a$ is unaffected.

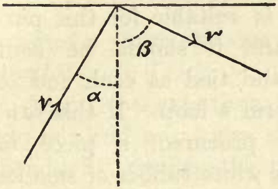


Fig. 207.

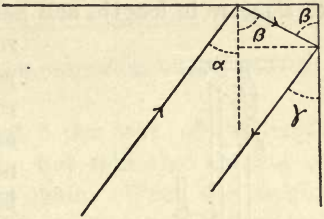


Fig. 208.

Hence (a) $v' \sin \beta = v \sin a.$

Perpendicular to the cushion we have

(b) $v' \cos \beta = -ev \cos a.$

Dividing (b) by (a) we have

$$\cot \beta = -e \cot a.$$

If the ball strikes another cushion at right angles to the first, we have

$$\tan \gamma = -e \tan \beta = \frac{-e}{\cot \beta}$$

$$= \frac{-e}{-e \cot a} = \tan a.$$

Hence $\gamma = a$, and the ball after striking two cushions is moving parallel to its original direction.

§ **137. Stress and strain. Young's modulus.** In the following paragraphs we do not propose to do more than indicate the kind of measurements by which information is obtained about the strength of materials. For fuller details an engineering or technological textbook should be consulted, such as Perry's *Applied Mechanics.*

§ **138. Strain.** Take an india-rubber rod, half a metre to a metre in length, and half an inch in diameter. White rubber is suitable for this purpose, and it should be bent round and tied at each end so as to form a loop. If this cannot be procured, a piece of black or white rubber of smaller diameter may serve. Hang one loop over a stout hook and hook a scale-pan on to the other end. Add enough weights to keep the rod taut. Now stick two needles through the rubber near the ends and fix a metre scale upright so that the points of the needles are near the graduations of the scale (Fig. 209). Take the readings of the pointers. The difference is the length of rubber of which the extension is to be measured. Keep adding small weights and reading the pointers after each addition, taking care to allow the same time to elapse between adding the weight and taking the reading in every case. The following

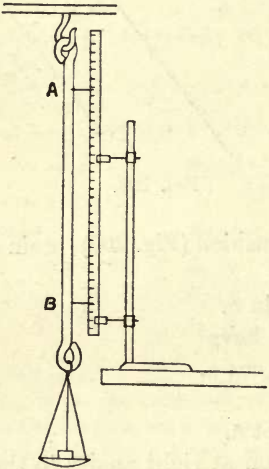


Fig. 209.

is a series of actual readings taken with a rubber rod as described:—

Load.	A.	B.	Difference.	Extension.
Scale-pan.	3 in.	31·2 in.	28·2 in.	0 in.
+ 4 oz.	3 „	31·4 „	28·4 „	0·2 „
+ 8 oz.	3·05 „	31·6 „	28·55 „	0·35 „
+ 12 oz.	3·1 „	31·9 „	28·8 „	0·6 „
+ 1 lb.	3·12 „	32·1 „	28·98 „	0·78 „
+ $1\frac{1}{2}$ lb.	3·2 „	32·6 „	29·4 „	1·2 „

Plot a curve showing how the extension varies with the load.

When the extension is about 5 per cent. of the length do not add any more weights; but take the weights off gradually, reading the pointers again. When the weights have all been removed, wait a few minutes, and then look to see if the pointers have gone back to their original position. Plot the values of the extensions for the unloading, along with those obtained for the loading. Do the two sets of values coincide ?

Repeat the experiment, waiting longer before taking each reading. Do you find that the extensions are increased ?

Measure the diameter of the rubber rod, and record it for future calculations.

§ **139. Extension of a metal wire.** For this purpose a vernier depth gauge will be found useful. The vernier should be screwed on to the wire, which may be a copper wire of No. 20, S.W.G., and the scale piece is also screwed on to another wire hanging from the same support. The arrangement has the advantage that any yielding of the support from which the wires hang does not affect the accuracy of the measurements. The following is a set of readings taken with an arrangement of this kind :—

Length of wire, 3 metres. Diameter, 1·2 mm.

Load.	Vernier Reading.	Extension.
Scale-pan.	0·57 cm.	0
+ 1 lb.	0·58 „	0·01 cm.
+ 3 lb.	0·60 „	0·03 „
+ 10 lb.	0·67 „	0·10 „
+ 14 lb.	0·70 „	0·13 „
+ 1 lb.	0·58 „	0·01 „
+ 28 lb.	0·84 „	0·27 „
+ 42 lb.	0·99 „	0·42 „
+ 56 lb.	1·15 „	0·58 „
+ 84 lb.	1·45 „	0·88 „
+ 1 lb.	0·58 „	0·02 „
+ 98 lb.	1·62 „	1·05 „
+ 112 lb.	1·86 „	1·29 „
+ 0	0·74 „	0·17 „

Now on examining these readings it is obvious that after a load of 112 lb. had been applied, the wire did not return to its original length when the load was taken off.

On plotting the loads and extensions, we find that after 98 lb. the 'curve' ceases to be straight. The elastic limit has been reached, and the wire has taken a permanent set. With india-rubber the same thing may be noticed, as well as the gradual yielding when the load is left on.

Similar sets of readings may be taken for iron wire, and the elastic limit found. Steel pianoforte wire may also be tried, with larger loads.

§ **140. Hooke's Law.** From a series of experiments of this kind, the student will be prepared to accept Hooke's Law—'Ut tensio sic vis.'

The extension is proportional to the stretching force. It is obvious that if the stretching is uniformly distributed, the extension of any length of wire is proportional to that length. So we reckon the **strain** as the extension for unit length: and this is obtained by dividing the extension by the length. Thus if a wire 20 cm. in length becomes 20·5 cm. long, the **strain** is $\frac{0\cdot5}{20}(=\cdot025)$.

The **stress** is the force per unit area of cross-section. Thus if a wire of 0·05 in. diameter is stretched by a force of 10 lb. weight, the stress is 10 lb. for an area of $\frac{3\cdot14\times(\cdot05)^2}{4}$ sq. in., which is $\frac{4\times10}{3\cdot14\times(\cdot05)^2}=5095$ lb. to the sq. in.

Now since the strain is $\frac{\text{extension}}{\text{original length}}$, and the stress is $\frac{\text{load}}{\text{area of cross-section}}$, it is plain that '**Strain is proportional to stress**' is another way of putting Hooke's law.

The fraction $\frac{\text{stress}}{\text{strain}}$ is called **Young's modulus,** and depends only on the nature of the material.

Example. A wire of diameter 0·12 cm. and 4 metres in length is stretched by a weight of 10 kilogrammes. The extension is 3 mm. Find Young's modulus (in dynes per square centimetre) for the material of which the wire is composed.

$$\text{Stress}=\frac{10000\text{ grammes}}{\pi\times(\cdot06)^2\text{ sq. cm.}}=\frac{9810000}{3\cdot14\times\cdot0036}\text{ dynes per sq. cm.}$$

$$\text{Strain}=\frac{0\cdot3}{400}=\cdot00075.$$

$$\therefore \quad \frac{\text{stress}}{\text{strain}} = \frac{9810000}{3{\cdot}14 \times {\cdot}0036 \times {\cdot}00075} = 1{\cdot}15 \times 10^{12} \text{ dynes per}$$

sq. cm. = 17000000 lb. per sq. in. nearly.

(i) Find Young's modulus for rubber, also for copper, from the readings given in §§ 138, 139.

Putting the foregoing results into algebraical language we have

$$\frac{T}{E} = \frac{l' - l}{l},$$

where T = the force per unit area required to stretch a wire of length l to a length l', and E is Young's modulus.

§ 141. Pressures and tensions. When the stress is compressive instead of tensile, a pressure instead of a tension, Hooke's Law is still true within the limits of elasticity, if we measure the diminution of length instead of the extension, and call the ratio of this diminution to the original length the strain. It is found that Young's modulus is the same for the same material whether that material is subjected to a push or a pull, whether it acts as a tie or a strut. (§ 58.)

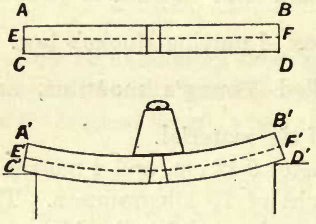


Fig. 210.

§ 142. Bending of a beam. Take a bar (india-rubber, which can be visibly bent, is suitable) and mark on it some parallel vertical lines. Now support it as in Fig. 210 and load it. The line AB will be compressed and become $A'B'$, while CD will be extended, as $C'D'$. There will be one line, EF, however, which remains the same length. All points in the beam above this **neutral line** are being compressed, while all parts below are being stretched. In each case the strain is obviously proportional to the distance above or below the neutral line.

Of course the distortion shown in Fig. 210 is much greater than occurs in the average beam or girder: but it is useful to examine it as illustrating what happens whenever a beam is loaded.

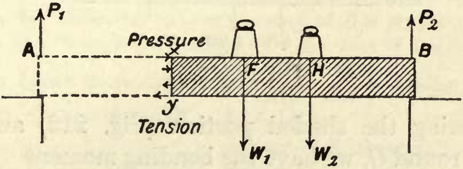


Fig. 211.

If we consider the equilibrium of the shaded portion of the beam in Fig. 211, it is plain that the pressure and tension of the unshaded portion exert a couple in a clockwise direction, and this couple must be equal and opposite to the couple $P \times BX - W_1 \times FX - W_2 \times HX$, &c. This couple, balanced by the molecular forces of the particles of the beam at XY, is called the **bending moment** at xy.

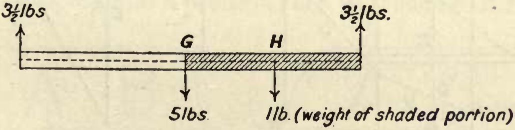


Fig. 212.

Example. A uniform beam, 4 ft. long, weighing 2 lb. is loaded by a weight of 5 lb. placed in the middle, and is supported at each end. Find the bending moment (*a*) in the middle, (*b*) 1 ft. from the end.

Obviously the reactions at the supports are, each of them $3\frac{1}{2}$ lb.

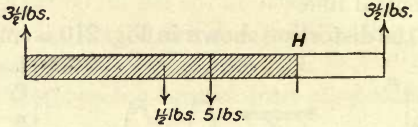

$3\frac{1}{2}$ lbs.
$3\frac{1}{2}$ lbs.
H
$1\frac{1}{2}$ lbs. 5 lbs.

Fig. 213.

Considering the shaded portion (Fig. 212) and taking moments round G, we have the bending moment

$$= 3\tfrac{1}{2} \text{ lb.} \times 2 \text{ ft.} - \text{weight of shaded portion} \times 1 \text{ ft.}$$
$$= 7 - 1 \text{ (foot-pound units).}$$
$$= 6 \text{ (foot-pound units).}$$

The bending moment at $H = 3\frac{1}{2} \times 1 - \frac{1}{2} \times \frac{1}{2}$

$$= 3\tfrac{1}{4}.$$

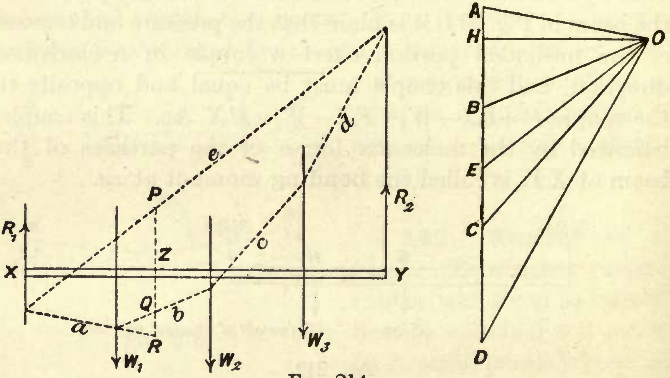

A
H
O
B
E
C
D
d
e
P
Z
c
R_2
R_1
X
Y
Q
b
a
R
W_1
W_2
W_3

Fig. 214.

Or, taking the equilibrium of the shaded part in Fig. 213, we have

$$\text{Moment round } H = 3\tfrac{1}{2} \times 3 - 1\tfrac{1}{2} \times 1\tfrac{1}{2} - 5 \times 1.$$
$$= 3\tfrac{1}{4}.$$

§ **143. Bending moment by graphic method.** The bending moment may also be found graphically (Fig. 214) by means of the link polygon (§ 95).

For if *ABCD* is the unclosed force polygon, then by drawing links, starting from a point in W_1, parallel to *OA, OB, OC, OD*, viz. *a, b, c, d*, we find that the closing link is *e*. Draw *OE* parallel to *E*; then *DE, EA* give us R_1, R_2. Now the bending moment at *Z* is proportional to *PQ*, the vertical line through *P* terminated by the sides of the link polygon. For if *OH* is drawn horizontal we have $\frac{PR}{XZ}=\frac{AE}{OH}$ by similar triangles.

$$\therefore\ PR\times OH = XZ\times AE$$

$$= \text{moment of } R_1 \text{ about } Z.$$

Similarly $$QR\times OH = \text{moment of } W_1 \text{ about } Z$$

$$\therefore\ PQ\times OH = \text{bending moment.}$$

§ **144. Shear.** If we again consider the equilibrium of the shaded portion of the beam in Fig. 211, taking the vertical forces only, it is obvious that there must be an upward force $(W_1+W_2-P_2)$ at *xy* to keep the others in equilibrium. There is a force which prevents the shaded portion slipping into a position such as is shown in Fig. 215.

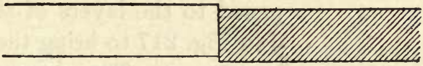


FIG. 215.

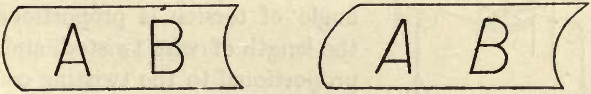


FIG. 216.

A stress tending to make the particles of one layer slide over the next layer is called a **shear**, and the kind of strain which it produces is illustrated in Fig. 216, which shows the leaves of a book acted on by a shearing force. When holes are punched in a metal plate, it is a shearing force which has to be used to make the hole.

(i) A beam 10 ft. long is supported at both ends and has loads of 14 lb. placed along it at intervals of 2 ft. Find the bending moment and shearing force in the middle. Neglect the weight of the beam.

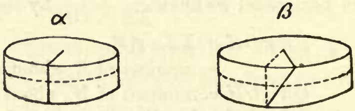


Fig. 217.

§ 145. Torsion. When a wire is twisted it is subjected to a shearing stress. This is obvious if we consider what must have happened to the layers of the cylinder *a* in Fig. 217 to bring them into the position β. The amount of twist or torsion is measured by the angle through which any radius of the wire has been turned (Fig. 217, β). An apparatus of the kind shown in Fig. 218 shows readily that the angle of torsion is proportional to the length of wire twisted, and also proportional to the twisting couple. One end of the wire must be firmly clamped, and the other firmly fixed in a pulley which can be turned by a couple supplied by two equal weights. The amount of twist for different lengths can be read by pointers attached to the wire.

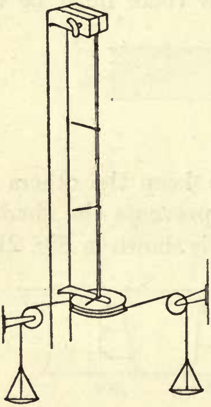


Fig. 218.

The second of the above laws is applied in the torsion balance, where the repulsion of an electric charge on another charge is balanced by twisting a fibre supporting one of the charged bodies, and bringing it back to its first position. The angle of twist is measured on the graduated circle at the top.

CHAPTER XVIII

LIQUIDS

§ 146. Different behaviour of materials under strain. In Fig. 219 the four equal forces F_1, F_2, F_3, F_4 can be regarded, if taken in pairs (F_1, F_2) and (F_3, F_4), as forming the two Forces P, P. Or again, taking them in pairs (F_1, F_4) (F_2, F_3) they constitute another pair T, T. Or we may take them as (F_1, F_3) (F_2, F_4); in which case they form two shearing stresses. It is plain then that if the shaded area is subjected to a pressure such as P, P, it will be pushed out of shape unless it is capable of resisting a tension such as T, T, and also a shearing stress.

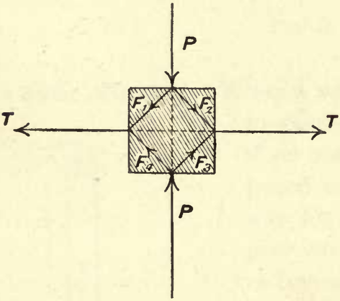


FIG. 219.

(Of course the actual problem deals with a *volume*, not with an area. The shaded portion should represent a cube, rather than a square, and stresses T, T perpendicular to the plane of the paper are created.)

It will be useful to consider the behaviour of a large number of different substances when subjected to a pressure, as for instance, when a block of the substance is laid on a flat table and a weight laid on top of it.

Solids offer resistance to the flattening process in more or less degree; e. g. **steel, lead** (which can be pressed out of shape by a very heavy weight), **india-rubber**, of which the elasticity or tendency to recover its shape, though not nearly so great as that of steel, is more noticeable owing to its greater range, **wax, putty.** In some of these the resistance offered to a shearing stress may be smaller than that offered to a tension. An **apple** can be crushed by a weight, and it can be twisted in half easily by a shearing force, but it offers considerable resistance to a simple pull. **Sand**, on the other hand, which is merely a collection of loose solid particles, offers more resistance to shear than to tension. Ropes of sand are proverbial; but sand can be heaped up and weights can be put on it, owing to the fact that the particles do not slide over one another without friction, and therefore there is a resistance to shearing stress. **Treacle** and **honey** are substances which yield to any pressure, and yet offer a small resistance to tension. Yet they would be classed as **liquids**, and this brings us to the definition of a liquid as **a substance which offers no resistance to shearing stress.**

§ **147. Liquids, Pascal's Law.** There may be friction between the particles of a liquid in motion, and indeed there is. But there is no statical friction (§ 112), and consequently the force across any section of a liquid must be at right angles to the surface.

The above definition of a liquid would include gases, such as air. But there is this great difference between liquids and gases, that gases are almost infinitely compressible and possess elasticity of volume, so that energy can be stored up in a gas as in the spring of a watch. The air-gun is charged by compressing air and storing up potential energy, which is given back when the trigger is pulled. A liquid is to all intents and purposes incompressible, so that energy cannot be stored up in it. Heat may be developed in a liquid by stirring it up and creating friction. But it needs nearly 1,400 foot-pounds of work to raise 1 lb. of water 1°C. The friction of water is so slight that it is too small to be noticed in slow movements, and a vessel of water with two tightly fitting pistons (Fig. 220) may be regarded as a frictionless machine. Any work put in at *A* is given out at *B* undiminished. If *A* and *B* are of the same cross-section, and we regard the water as incompressible so that no energy can be stored up in it, we see that *B* must be driven out the same distance that *A* is pushed in. And since this distance multiplied by the force of the push gives the work done, the force at *B* must be the same as at *A*. This leads at once to Pascal's principle that any pressure applied to any square inch or square centimetre of the surface of a liquid is transmitted unaltered across every square inch or square centimetre in the liquid.

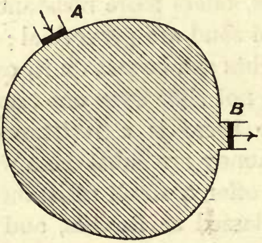


FIG. 220.

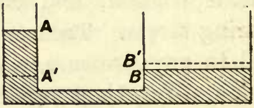


FIG. 221.

If the areas of the two pistons (Fig. 221) are *a* and *b* square inches, and if a pressure of *p* lb. to the square inch be given to *A*, the total pressure on *A* is *ap* lb., on *B bp* lb.

How far will B be forced up if A is forced down a distance AA^1 of x inches ?

§ **148. Equilibrium of a liquid.** When a liquid is at rest, every part of it is at rest. If a cup of tea or a beaker of water is stirred, the liquid can be set in motion and will go on revolving in the vessel until the friction between the particles of the liquid gradually brings it to rest. The friction is sufficient to do this ; but it is not sufficient to resist the constant force of gravity. If the water is muddy and we stir up the mud and leave it, in time the mud will settle to the bottom. The friction is not enough to hold up even the small particles of mud for any length of time. Therefore in considering the equilibrium of any portion of a liquid, we may neglect friction altogether.

Let A be any irregularly shaped region in a liquid. A has weight, which we may suppose to act at its centre of gravity. The only other force acting on A is the pressure of the surrounding liquid, which is, at every point of the surface, **normal,** i.e. at right angles to the surface. The arrows in Fig. 222 indicate the direction of the pressure at a few points. The resultant of all these pressures must be a force equal and opposite to the weight. Otherwise A would not be at rest.

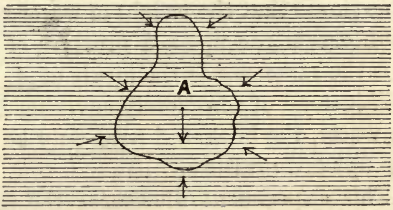


Fig. 222.

The resultant pressure of a liquid on any surface such as A is equal to the weight of liquid that would fill the space occupied by A.

Is it possible to verify this experimentally?

Take a small cylindrical bucket *A*, and a solid metal cylinder *B* that fills it exactly.

Remove one of the scale-pans of a balance and hang the bucket and cylinder from the hook (Fig. 223).

Counterpoise *A* and *B* by weights or shot in the other scale-pan. Now if *B* hangs below *A* instead of inside *A*, the balance should still be true. But if *B* is surrounded by water, the upward pressure of the water will at once force up the balance on that side. Equilibrium will be restored if water is poured into the bucket *A* so as to fill it exactly, *B* being at the same time completely covered as in the figure. This shows that the weight of water required to fill *A* exactly counteracts the upward pressure of the water on *B*. But *B* is the same size as the inside of *A*.

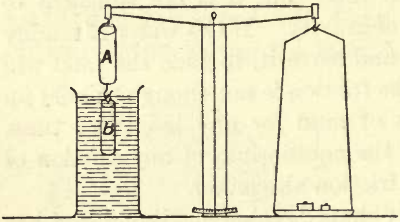


FIG. 223.

If instead of completely covering *B* it is only half covered by water, the bucket *A* need only be half full to counteract the upward pressure.

This experiment illustrates the principle, discovered by Archimedes, that the upward pressure of a liquid on any body is equal to the weight of liquid displaced by that body. This principle is concerned with the equilibrium of floating bodies and bodies immersed in a liquid, to be considered later. Here we are only considering the equilibrium of any portion of the liquid itself, which is held in position by the pressure of the surrounding liquid counteracting the force of gravity.

§ **149. Pressure the same at all points in the same horizontal plane.** Consider a small horizontal cylinder of liquid *AB* (Fig. 224).

The pressures at *A* and *B* are horizontal, and are at right angles to all the other forces acting on the cylinder, gravity included. Therefore the pressure at *A* must be equal to the pressure at *B*.

Now suppose one end *A* to be vertical, the other end *B* to make an angle a with the vertical. The pressure on *A* is horizontal; the pressure on *B* makes an angle a with the horizontal (Fig. 225).

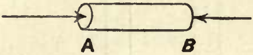


Fig. 224.

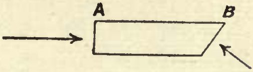


Fig. 225.

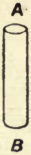


Fig. 226.

Pressure on A = horizontal component of pressure on B.
= pressure on $B \times \cos a$.

Now if p_1 = pressure on unit area at A,
p_2 = ,, ,, ,, B.

Then $p_1 \times$ area of $A = p_2 \times$ area of $B \times \cos a$; but area of A = area of $B \times \cos a$.

$$\therefore \quad p_1 = p_2.$$

Hence the pressure on unit area at the same level is the same in all directions.

§ **150. Pressure increases with the depth.** If *AB* is a vertical cylinder of liquid, it is plain that the pressure at *B* must be greater than the pressure at *A* by the weight of the cylinder itself. The weight of this cylinder is obviously proportional to its length (Fig. 226).

If A is at the surface of the liquid the pressure at B is equal to the surface-pressure + weight of column AB. Let the unit of area be 1 sq. cm. and the depth AB 10 cm. Then pressure per sq. cm. at B = surface-pressure + weight of 10 cubic centimetres of the liquid.

Now in Fig. 227, if B and B' are in the same horizontal plane, the pressures[1] at B and B' are the same. And the depth AB will be the same as the depth $A'B'$ if the pressure of the air at A is the same as the pressure at A'.

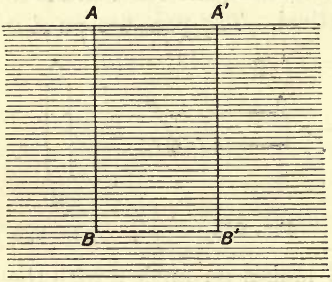


FIG. 227.

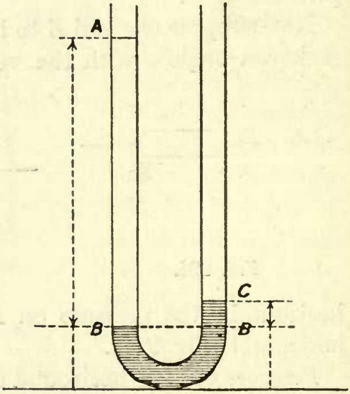


FIG. 228.

Pour water into a U-tube and measure carefully the height of each surface above the table. Now alter the pressure on one surface by sucking or blowing into that branch of the tube. Repeat the experiment, using mercury instead of water, taking care that you do not get the mercury into your mouth.

Find whether the pressure of the gas in the pipes is the same as the pressure of the air. To do this connect one branch of the U-tube by a piece of rubber tubing to a gas jet, and turn the gas on full. Measure carefully the difference in level of the water in the two branches. By how much does the pressure of the gas exceed the pressure of the air ? Give your result in grammes weight per sq. centimetre.

---

[1] The word *pressure* is frequently used alone to mean pressure on unit area. Thus 'a steam-pressure of 90 lb.' means 90 lb. to the square inch.

§ **151. To compare the heights of two different liquids which do not mix.** Pour some mercury into a U-tube. Then pour water into one branch so that the depth of water *AB* (Fig. 228) is at least 10 in. The mercury will rise in the other branch. Measure the heights of *A*, *B*, *C* above the table to the nearest millimetre, and so find the heights of the columns *AB*, *CB* which balance each other, since *B*, *B* must be at the same pressure. Find by calculation what height of water is equivalent to 1 centimetre of mercury. In this we assume that the pressure of the air is the same at *A* and at *C*. Is this justifiable ?

Keep your result to compare with that obtained from the specific gravity bottle, using mercury and water (§ 160).

CHAPTER XIX

ATMOSPHERIC PRESSURE

§ **152. To measure the pressure of the air.** When the air is sucked out of one branch of a U-tube containing mercury, the difference in level of the two surfaces measures the difference in the pressure of the air. If we could suck or pump out all the air on one side, the pressure of the air on the other would be balanced by this column of mercury alone. Mercury being 13·6 times as heavy as water, it follows that water can be sucked or pumped to a height 13·6 times that to which we can pump mercury. To find exactly what this height is, a tube a yard long, closed at one end, is filled with mercury, all air bubbles being expelled; it is then closed with the finger and inverted in a trough of mercury, when the finger can be removed. The mercury in the tube descends a little. Measure the difference in height between the inner and outer surfaces of the mercury.

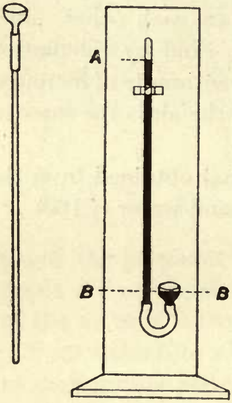


Fig. 229.

A convenient form of this apparatus is shown in Fig. 229. The open end of the tube is connected by a piece of pressure-tubing to a glass funnel. Mercury is poured in till it fills the long glass tube and the rubber-tubing, and the long tube is then turned up into the second position, when it can be fastened to an upright board. Measure the difference between the heights of A and B.

What is there above A?

What keeps the mercury up to A?

What height of mercury is equivalent to the pressure of the air?

What height of water?

A cubic centimetre of water weighs 1 gramme. Mercury is 13·6 times as heavy as water. What is the weight of a column of mercury 76 cm. high and 1 sq. cm. in section?

What is the pressure of the air in grammes weight per sq. cm.?

1 lb. = 453·6 grammes.
1 sq. in. = 6·45 sq. cm.

Find the pressure of the air in lb. weight to the square inch.

This instrument for measuring atmospheric pressure, invented by Torricelli, a pupil of Galileo, is called a barometer. The space above the mercury in the closed tube is called the Torricellian vacuum.

§ **153. Weight of the air.** A flask is fitted with a cork having a short glass tube through it, and the glass tube has a piece of rubber-tubing with a screw-top attached to it, by which it can be connected to an air-pump (see Shenstone's *Chemistry*, p. 71). The air is pumped out of the flask, the tap is screwed up, and detached from the air-pump. The flask is then weighed, the tap is unscrewed, and the increase of weight, due to the air which rushes in, is found.

The weight of a litre of air at ordinary pressure is about

1·29 gramme. How high must a column of air of this density be to balance a column of mercury 76 cm. in height?

(Suppose the two columns to be each 1 sq. cm. in section. The weight of the mercury column is $13·6 \times 76$ grammes. The weight of the air column is $·00129 \times h$, where h is the height in centimetres.)

§ **154. Probable height of the atmosphere.** If the air were of the same density throughout, the weight of the air column considered in the previous section would correspond to a height of about 5 miles. But we know that the atmosphere extends far beyond this, and various considerations lead us to the conclusion that the atmosphere must extend to at least 200 miles from the earth's surface. As we go up higher and the weight of air above gets less, the density of the air also diminishes, and the atmosphere is thinner or rarer. We have now to find the connexion between the change in the density and the change in the pressure.

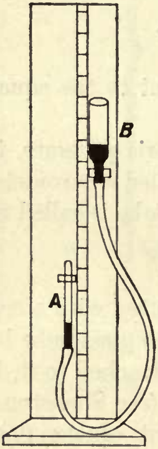


Fig. 230.

§ **155. Boyle's Law.** In 1662 Robert Boyle, son of the great Earl of Cork, and one of the founders of the Royal Society, published his 'Doctrine concerning the weight and spring of the air.' For an account of Boyle's experiments in his own words the reader may consult Tait's *Properties of Matter.* The apparatus which is to be described is essentially the same as that used by Boyle. *A* is a straight tube, graduated for equal volumes, closed at the top and connected by a long piece of pressure-tubing containing mercury to the moveable cistern *B*, which can be raised or lowered. The air in *A* is at the same pressure as the air outside when the mercury in *B* is at the same level as that in *A*. Read the volume of the air in *A* when this is the case. Also read the height of the barometer, which gives the pressure of the air in inches (or centimetres) of mercury. Now raise *B*. The air in *A* will be compressed and the mercury will rise slightly in *A*. Read carefully the *difference* in height between the two surfaces. This added on to the barometric height gives the new pressure on the air in *A*. At the same time the volume of the air should be read carefully. Take a number of readings at different positions of *B*, and tabulate your results thus:–

Volume of air in *A*.	Difference in level.	Barometric height.	Pressure in cm. of mercury.	Volume × pressure.
20 cc.	0	75 cm.	75	1500
16·5 cc.	16·2 cm.	,,	91·2	1505

What conclusions do you draw from an examination of the results in your last column, which is the product of the numbers expressing the volume and the pressure? What does the volume of the imprisoned air become when the cistern *B* is pulled up high enough to make the difference in the mercury levels equal to the barometric height, i.e. to double the pressure?

Plot a curve from your results, taking volumes as abscissæ and the reciprocals of the pressures as ordinates.

§ 156. Boyle's Law continued. Diminished pressure. A straight graduated tube *AB* (Fig. 231), with a tap at *A*, but open at *B*, is lowered into a deep vessel containing mercury, the tap being open. The mercury rises to the same level inside the tube as outside. The volume of air *AC* is noted and the tap is then closed. *AB* is now raised and the air inside the tube expands. Note the volume *AC′*, also the difference in level (*CC′*) between the mercury inside and outside the tube. This represents the difference in pressure between the air inside and the air outside. Find the pressure of the air *AC′* by subtracting *CC′* from the height of the barometer, and tabulate your results as in § 155, taking several observations.

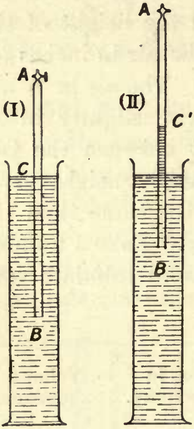


FIG. 231.

Plot a curve as before. N.B. The apparatus of Fig. 230 can be used for diminished pressure, if *A* can be raised above *B*.

Boyle's Law. If the student has made the preceding experiments carefully he will be in a position to accept the following statement of what is known as Boyle's Law:—

The volume of a given mass of air (or any other gas) varies inversely as the pressure on it, if the temperature meantime remains constant.

This may be stated thus: $vp =$ a constant, when t is constant. When a gas is near its liquefying point, it does not obey Boyle's Law. A gas that obeyed the law strictly, as well as the other law as to expansion with rise of temperature known as Charles' Law, would be called a perfect gas. Air and hydrogen at ordinary temperatures may be regarded as perfect gases.

§ **157. Examples.** (i) What is the weight of a litre of air collected at the top of a mountain where the barometer stands at 50 cm. ? (1 litre of air at a pressure of 76 cm. weighs 1·29 grammes.) A litre of air at 50 cm. pressure would contract if the pressure were increased to 76 cm., and would only occupy $\frac{50}{76}$ of a litre. Its weight would therefore be

$$\tfrac{50}{76} \times 1{\cdot}29 \text{ grammes} = 0{\cdot}85 \text{ gramme nearly.}$$

(ii) A barometer tube is partly filled with mercury, leaving 10 cm. of the tube, which is then closed with the finger and inverted in a trough of mercury. The air bubble expands to three times its original size, i.e. there is a space of 30 cm. above the mercury in the tube. How high will the surface of this mercury be above the mercury in the trough, the height of the barometer being 75 cm. ?

The volume has increased in the ratio 3 : 1.

Hence the pressure has diminished in the ratio 1 : 3.

But the air was at first at atmospheric pressure, i.e. 75 cm. of mercury.

∴ the new pressure is equivalent to 25 cm. of mercury, and therefore the mercury in the tube must be 25 cm. lower than that in the barometer, which is at 75 cm.

∴ required height = 50 cm.

(iii) A flask containing air is forced neck downwards under water. How deep must it be forced before the water half fills the flask ? (Height of water barometer = 34 ft.)

(iv) A flask holding 350 cc. is forced neck downwards under water to a depth of a foot. How much water will have entered ?

(v) A diving-bell is lowered to a depth of 17 ft. in water. A person in the diving-bell corks up a bottle of air, the diameter of the neck of the bottle being 1 in. What force is required to hold the cork in when the bottle is brought up to the surface ?

(vi) How and why is the height of a barometer affected in ascending a mountain? Why is the effect of the first thousand feet of ascent upon the barometric height greater than that of the second thousand? (*Lond. Matric.*)

(vii) Draw a section of a common suction-pump and explain its action.

The spout of such a pump being 24 feet above the surface of the water in the well, the area of the piston 4 square inches, and the mechanical advantage of the lever handle 5, find the force necessary to work the pump, and the work done in pumping ten gallons of water. (A gallon of water weighs 10 lb., and a cubic foot 62·4 lb.) (*Lond. Matric.*)

(viii) Explain the action of a force-pump.

The sectional area of the piston of a force-pump being 8 square inches, find the force necessary to pump water from a reservoir 10 feet below the pump to a cistern 40 feet above it. (*Lond. Matric.*)

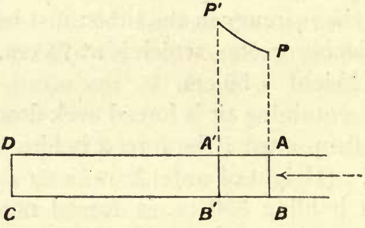


FIG. 232.

§ **158. Work done in compressing a gas.** Imagine some air contained in a cylinder *ABCD* to be compressed by forcing a piston from the position *AB* to *A′B′*. If the pressure were constant, then this pressure multiplied by the distance AA' would give the work done. If AA' is very small, the pressure may be taken as constant for that small change in volume. If we draw perpendiculars AP, $A'P'$, &c., to represent the gradually increasing pressures, then the area $PAA'P'$ represents the work done in the compression (Fig. 232).

A gas differs from a liquid in being compressible. Work done in compressing the gas is partly stored up as potential energy and is given back when we allow the gas to expand again. But a liquid being incompressible cannot have energy stored up in it in this way.

CHAPTER XX

SPECIFIC GRAVITY

§ **159. Density and specific gravity.** The density of a substance is the number of grammes in one cubic centimetre of it (C.G.S. system) or the number of pounds in one cubic foot of it (English measure).

To find the density of a substance, measure the weight of a piece of it, and also its volume. Hence calculate the weight of unit volume.

Find the volume of a metal cylinder by measuring its length l and diameter d $\left(\text{volume} = \frac{\pi d^2 l}{4}\right)$. Check this measurement by finding how much water the cylinder displaces. This can be done by putting the cylinder in a narrow glass with a mark above the top of the cylinder. Fill with water up to this mark. Then take the cylinder out with as little water as possible on it, and measure out from a burette enough water to fill the glass up to the mark again. Remember that to read the true surface of water your eye must be on a level with it, and the lowest line you see is the true surface.

The **specific gravity** of a substance is the number obtained by dividing the weight of any volume of it by the weight of an equal volume of water. Weigh a solid, the glass stopper of a bottle, for instance. Fill a measuring glass with water till it overflows at the spout. Weigh a small beaker. Hold the beaker under the spout, and lower the glass stopper slowly into the measuring glass, collecting the overflow in the beaker.

Weigh again.

Find the weight of water displaced and calculate the specific gravity of glass.

§ **160. Specific gravity bottle.** This is a small bottle with a close-fitting stopper which has a small hole drilled in it.

Weigh it empty. Then fill it with methylated spirit and push in the stopper, wiping off the spirit that overflows. Weigh again.

Then empty the bottle, rinse it out with water, fill it with water, insert the stopper, wipe, and weigh again.

Subtract the weight of the empty bottle from each of the other two, and thus obtain the weights of equal volumes of spirit and water.

Find the specific gravity of methylated spirit.

As 1 cc. of water weighs 1 gramme, you can find the volume of the contents of the bottle, and hence also the density of methylated spirit.

§ **161. Specific gravity of small fragments of a solid.** Take some small pieces of glass, or a number of shot. Weigh them (w_1). Fill the specific gravity bottle with water, wipe it, put it in the scale-pan along with the solid fragments, and weigh again (w_2). Count the fragments and put them all inside the bottle, getting rid of air bubbles and replacing the stopper. Wipe off the water that runs over and weigh again (w_3).

What is the weight of water displaced by the solid fragments?

What is their specific gravity?

§ 162. Principle of Archimedes.

The upward pressure of a liquid on a solid suspended or floating in it is equal to the weight of liquid displaced by the solid.

For an experiment verifying this see § 148.

In the case of a solid heavier than water, we can find the upward pressure by weighing, hanging first in air then in water. The difference gives the upward pressure of the water, i.e. the weight of water displaced.

If an ordinary balance is used, the solid must be hung by a piece of cotton from the hook above the scale-pan, and the water must be in a small beaker standing on a wooden bridge. The solid must be completely under the water, and not touching the side. The bridge must not touch the scale-pan.

A piece of lead weighs, when hanging by a small piece of cotton, 36·52 grammes, when hanging in water it weighs 33·29 grammes. Hence the weight of water displaced by the lead is 3·23 grammes.

The specific gravity of lead is therefore

$$\frac{36{\cdot}52\text{ g.}}{3{\cdot}23\text{ g.}} = 11{\cdot}3.$$

Note that 11·3 is a *number* merely.

§ 163. Specific gravity of a liquid by Archimedes' principle. Suppose that a solid weighs w_1 in the air, w_2 in water, and w_3 in methylated spirit.

Then weight of water displaced $= w_1 - w_2$,
weight of spirit displaced $= w_1 - w_3$.

But the volume of the spirit is the same as that of the water.

Therefore specific gravity of spirit $= \dfrac{w_1 - w_3}{w_1 - w_2}$.

§ 164. Specific gravity of a solid that floats in water. Weigh a piece of cork (w_1).

Tie a piece of lead to the cork so that the cork may be hanging in air and the lead under water, and weigh again (w_2).

Now tie the cork and lead together so that both are below the surface of the water, and weigh again (w_3).

What is $w_2 - w_3$?

What is the specific gravity of cork?

§ 165. Flotation. A test-tube loaded with shot is useful for flotation experiments. Weigh it. Lower it slowly into a brimming measuring glass until it floats, collect the overflow and weigh it as in § 159. The two weights should be the same. Repeat the experiment using methylated spirit instead of water.

If the end of a test-tube is flattened, the volume of the part of it below the surface can be calculated by measuring its length, marked by an elastic band on the tube at the surface of the water, and its diameter (average of three different measurements).

From these experiments the truth of the following statement may be inferred :–

The weight of a floating body is the same as that of the liquid it displaces.

Show that this is included in the statement of § 162.

Example. A test-tube floating in water has 18 c.c. below the surface; floating in brine only 17 c.c. are below the surface. What is the specific gravity of the brine?

Weight of test-tube = weight of 17 c.c. of brine;

= weight of 18 c.c. of water.

$\therefore$ weight of 1 c.c. of brine = weight of $\frac{18}{17}$ c.c. of water.

$\therefore$ specific gravity of brine = 1·06 nearly.

§ 166. Examples.

(i) What is the resultant of all the pressures which a liquid exerts on a solid completely immersed in it? Describe any experiment to prove the correctness of your statement. (*Lond. Matric.*)

(ii) A cork floats on the surface of water under the receiver of an air-pump. State what will be the effect of admitting air into the receiver, giving clear reasons for your answer. (*Lond. Matric.*)

(iii) Two masses, one of aluminium and the other of lead, when weighed by a perfectly adjusted balance, balance one another in air, but would no longer do so if the weighing were made in a vacuum. Explain this, and show which would appear to be the heavier in the vacuum. (*Lond. Matric.*)

(iv) A block of wood whose volume is 450 c.c. floats in water with 100 c.c. above the surface. How much water is displaced by the block? Thence find the weight of the block. (*Lond. Matric.*)

(v) Mercury is poured into a U-tube which has a section of one square inch, and which is standing with its limbs vertical, so as to more than occupy the bend; 20 cubic inches of water are then poured into one limb. Through what height will the mercury rise in the other? (Specific gravity of mercury = 13·6.)

Explain how such an arrangement as the above might be used to determine the specific gravity of a liquid. Is it necessary for this purpose that the two limbs should have the same sectional area? (*Lond. Matric.*)

(vi) A glass tube, 30 cm. long and $\frac{1}{2}$ sq. cm. in cross section, is closed at one end. Its weight is 4 grammes and 10 grammes of mercury are poured into it. To what depth will the tube sink in a liquid of specific gravity 1·2, and what is the specific gravity of a liquid in which it floats vertically with 2 cm. of its stem above the surface? (*Lond. Matric.*)

(vii) A wooden cylinder of uniform density floats in water with its axis vertical and one-third of it above the surface of the water. In another liquid it floats similarly, but with a quarter of its axis above the surface. Find the specific gravity of the liquid.

Why should the cylinder used for this experiment be short and thick rather than long and thin? (*Lond. Matric.*)

(viii) A beaker half full of water is placed on one pan of a balance and counterpoised by weights in the other pan, after which an iron cube of 4 inches edge suspended by a cord is let down into the water without being allowed to touch the bottom of the beaker. What effect is produced upon the balance, and what adjustment of weights must be made to restore equilibrium? How would the result have been affected if spirit had been used instead of water? (The specific gravity of spirit is 0·9 and that of iron 7·5.) (*Lond. Matric.*)

(ix) Explain what is meant by the density of a substance, and describe any one method by which you could find the density of a piece of coal. (*Lond. Matric.*)

APPENDIX I

EXPERIMENTAL WORK

THIS may be divided into two sections: experiments done by individual students in the laboratory, and experiments performed before a number of students by a lecturer or demonstrator in a class-room or laboratory. Most of the experiments on motion come under the latter head, and the statical work will be found to gain very much if it is performed by the teacher in the class-room, and repeated by the students for themselves. If students are to begin by themselves in the laboratory, it will probably be found desirable that statics should come first. In this case Chapters VI to IX should be taken after Chapter I, and the following hints may be found useful.

1. Checking the accuracy of a spring-balance (§ 2).

2. Spiral spring of copper wire (§ 2). The spring should be coiled on a pencil, or a piece of straight glass tubing, and bent into a hook at each end, so that it may hang vertically from the ring of a retort stand, with a scale-pan hanging from it. If number 17 wire is used, weights increasing by 2 ounces, or 50 grammes, at a time may be used; and after each addition the whole weight should be removed to see if the spring returns to its former position.

3. Tension (§§ 3, 4). Tie a long string to a weight of 2 lb., pass it over a pulley, and tie the other end to the hook of a spring-balance. Pull the ring of the spring-balance so as to lift the weight, and take readings when the two parts of the string are parallel and at right angles. Does the pulley do anything more than alter the direction of the force ?

Hang two pulleys in the same horizontal line, attach strings to the hook and ring of a spring-balance, and let them pass over the pulleys and be tied to two equal weights, the spring-balance being kept horizontal by the pull of the weights. What is its reading ? Is the tension the same throughout the length of the string ? Try two spring-balances in series as in § 3.

4. Resultant of two forces in the same direction. Hang a weight from two spring-balances in parallel (§ 3). Is the sum of their readings equal to the weight ? Hang a string, with hooks at different places, from the hook of a spring-balance, and attach different weights to the hooks. Lift the balance by its ring. Is its reading equal to the sum of the weights ? Does it make any difference if you alter the order of the weights ?

5. Resultant of two forces in opposite directions. Put a 14 lb. or 7 lb. weight on the pan of a compression-spring-balance. Now hook a spring-balance to the weight and pull upwards, observing the readings of both balances. Does the reading of the lower one decrease by an amount equal to the reading of the upper ?

6. Parallelogram of forces (§§ 58–61). A strong framework with hooks from which pulleys may be hung is made by Cussons of Manchester, and will be found suitable for this and other experiments. Or light wooden pulleys, bushed so as to run freely on bradawls, can be fixed at any required point in a vertical board. Weights of $\frac{1}{10}$ lb. are useful for this work.

7. Inclined plane (§ 65).

8. Force required to pull suspended weight out of the vertical (§ 67, ii).

9. Polygon of forces (§ 70, ii).

10. Application of triangle of forces. Jib crane (§ 71).

11. Moments (§ 73).

12. Moments (*continued*) (§ 74).

13. Moments (*continued*) (§ 76). $\frac{1}{10}$ lb. weights are suitable.

14. Parallel forces. Two compression-spring-balances with their scale-pans removed and supporting a board loaded so as to bring their pointers to the zero point will be found suitable (Fig. 102 ; § 77, i, ii).

15. Parallel forces (*continued*) (§ 78). It will be found difficult to make the adjustments unless the weights and distances are first chosen so as to agree with the theory. See Fig. 106. The pulleys running on bradawls are particularly suited to this and the following experiment.

16. Couples (§§ 79, 80).

17. Centre of gravity (§ 85).

18. Centre of gravity. Special cases (§§ 86, 88).

19. To find to what height the kinetic energy of a moving body will carry it. Galileo's experiment (§ 99, i).

20. Friction experiments (§ 110).

21. Angle of friction (§ 111).

22. Efficiency of machines (§§ 116–119). A good type of experiment is that with the screw jack, but the following method applies to all. The velocity ratio, i.e. the ratio of the distance travelled by the scale-pan (Fig. 180) to the distance through which the weight is raised, must be measured experimentally. It can also be calculated. Let this ratio be N. Then if an effort P be applied, the theoretical load which it should lift is NP. If the actual load is W, then the loss $NP-W$ is the friction. Take a set of readings with different loads. Tabulate your results thus:—

Load W	Effort P	Theoretical Load NP	Friction $NP-W$	Efficiency $\frac{W}{NP}$

Plot graphs of (i) Load and Friction,
(ii) Load and Effort,
(iii) Load and Efficiency,

taking *loads* as abscissæ.

23. Stress and strain (§ 138).

24. Archimedes' principle (§ 148).

25. Pressure proportional to depth. Measurement of pressure (§ 150).

26. Comparison of heights of different liquids (§ 151).

27. Barometer (§ 152).

28. Boyle's law (§§ 155, 156).

29. Specific gravity and density (§ 159).

30. Specific gravity bottle. Liquids (§ 160).

31. Specific gravity of shot (§ 161).

32. Specific gravity by Archimedes' principle (§§ 163, 164).

33. Flotation (§ 165).

DEMONSTRATIONS

Hibbert's magnetic balance (Fig. 3) supplies a method illustrating the balancing of forces of different kinds. The repulsion of two like magnetic poles is balanced by the force of gravity. The apparatus is made by Messrs. W. G. Pye & Co., of Cambridge.

Gravity. Inclined plane experiments. There are various pieces of apparatus in the market for demonstrating the laws of uniformly accelerated motion. Atwood's machine, Henrici's and Barrell's apparatus, &c., are capable of giving good results. The electrical appliances for releasing the falling body and taking the time are sometimes found troublesome, and Galileo's more simple methods are more manageable. Steel motor-car bearings rolling down a polished groove in a thick board of oak or mahogany give good results. The steel balls are priced according to size; $\frac{3}{4}$ inch diameter is a suitable size (see §§ 31, 32, 34, 38).

Measurement of acceleration. Fletcher's apparatus, described in § 42, is supplied by Cussons of Manchester and Pye of Cambridge. A steel spring having a frequency of 10 vibrations per second may perhaps be found more useful than one having a frequency of 5.

Momentum experiments. Two trolleys (see § 53) can be made to illustrate the same laws as are demonstrated by Hicks's Ballistic Pendulum.

Mechanical equivalent of heat. Callendar's apparatus (§ 114) is supplied by the Cambridge Scientific Instrument Company.

APPARATUS

Spring-balances are supplied by Salter & Co., West Bromwich. Those reading $\frac{1}{4}$ oz. to 2 lb. are suitable for some experiments; others reading $\frac{1}{2}$ lb. to 25 lb. will also be useful. Compression-balances are supplied by the same firm. Salter's Family Balance, reading single ounces up to 14 lb., is a good type.

For weights, it is as well to use large weights in preference to small. Sets of 1 lb., 2 lb., 4 lb., 7 lb., 14 lb. are useful. For some of the experiments on Moments and Parallelogram of Forces, weight-holders and weights of $\frac{1}{10}$ lb. are suitable.

Many makers of physical apparatus supply material of this kind to technical institutes. A book like S. H. Wells' *Practical Mechanics* should be read by those intending to fit out a school laboratory.

Mr. G. Cussons, Technical Works, Lower Broughton, Manchester, and Messrs. W. G. Pye & Co., Granta Works, Cambridge, supply all the apparatus required for the course indicated in the preceding pages.

Pye's Little Jack is a useful form of levelling screw. A glass plate laid on three of these can easily be made to afford a horizontal surface (§ 60).

APPENDIX II

The following Exercises are taken from the questions in Practical Mathematics set in recent Army Examinations. Thanks are due to the Controller of His Majesty's Stationery Office for permission to make this selection.

1. Use the board with the attached pulley as an inclined plane, setting it at an angle of between 20° and 30° with the horizontal. Suspend suitable weights by a cord passing over the pulley and attached to the loaded trolley, so as to cause the trolley to travel slowly up the board when given a slight starting impulse. Observe the pull in the cord.

Also find the pull under which the trolley runs slowly down.

Assume that the average of these pulls is that under which the trolley would remain at rest if it were frictionless, and from this average and the measured inclination of the board, construct a force diagram and find the weight of the loaded trolley. Also find the resultant thrust of the cord on the pulley.

2. A cord supported at two points and containing a spring balance carries a mass hanging from a pulley. See that the mass is in its steadiest position, then note the inclination of the cord to the horizontal, and the pull in the cord. Repeat with different inclinations. Show your results in a table and in a graph. Give also the label-number of the mass.

From your graph give the pull in the cord when the inclination is 45°, and the inclination when the pull is 4·4 kilos. Test by observation.

3. Select A and B, two points about three feet above the table and $3\frac{1}{2}$ ft. apart. Fix the given pulley at the point A and tie the end of a cord at B. Tie the other end of the cord to a weight W on the table, making the cord BW $3\frac{1}{2}$ ft. long. Tie to the weight another cord which is passed over the pulley and fastened at a convenient place to the hook of a spring balance. By pulling downward on the spring balance the weight W can be lifted from the table, and at the same time the necessary pull measured. It will be found that the pull P required to hold W stationary in the air is different for every value of the length l of cord between the pulley and the hanging weight.

Make six observations of P for six different positions of the weight W and measure in each position the length l of cord between the pulley and W.

Draw a triangle of forces for each position recorded and thence get the value of the tension T of the cord BW.

Tabulate corresponding values of the ratio of T to W, and of the ratio of l to double the distance AB.

4. Weigh the given plate, and, in any way you please, draw a full-sized copy of it. Suspend the plate on a knife-edge, with a weight of 2 kilos hung from another hole; select with judgment the two holes to use. On your copy mark the point of suspension and the line of action of the 2-kilo weight. Transfer the 2-kilo weight to another hole, and mark on your copy its new line of action. From the observations you have made, determine and show on your copy the centre of gravity of the plate. Check your result by letting the plate hang freely and showing on your copy the line which is then vertical.

5. Arrange the bar *ABC* and the metre rule *DEF* as shown, so that when both are held horizontally the three threads *AH*, *CD*, *EK* may be all vertical. *D* and *E* are loops of thread tied tightly round the metre rule at the graduations 10 cm. and 25 cm. respectively. From *B* suspend the load *W* and note the distance, *y* cm., from *E* of the rider *R* when it is placed so as to balance the system. Tabulate the values of *y* corresponding to the following values of *W* :—

0, 100, 200, 300, 400, 500 gm.

Show in a graph how *y* depends upon *W*, and account for the fact that the graph is a straight line.

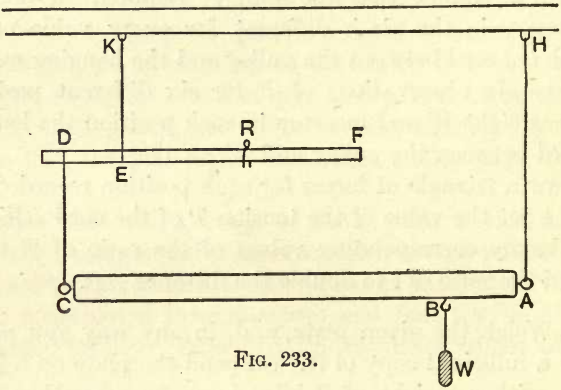


FIG. 233.

6. Weigh the given sheet of metal and find the position of its centre of gravity. Attach any three known loads to it by strings, one hanging vertically and the other two passing over pulleys. Tap the sheet to bring it to the steadiest position. You now have the sheet of metal in equilibrium under 4 forces. Mark these forces in direction and magnitude on the sheet of metal, and transfer them to drawing paper.

In any way you please find the horizontal and the vertical component of each force. What ought to be the value of the sum of the horizontal components ? and the sum of the vertical components ? and the moment of the forces about the centre of gravity of the sheet ? See to what degree of accuracy they have these values.

Also, supposing the weight of the sheet the only force known in magnitude, find from your drawing the magnitudes of the other 3 forces, and tabulate them beside the weights of the 3 loads that cause them.

7. Observe the suspended metal plate, and note the hole by which it is suspended. Find the centre of gravity of the metal plate.

Suspend the plate in its original position. To the hole at the narrow end fasten a cord, pass the cord over the fixed pulley and attach a weight of 400 grammes to its free end. Find, by equating moments, the weight of the plate. Sketch the arrangement.

8. Take a cord about 8 ft. long and tie it to two points *A* and *B* at the same level about 5 ft. apart so that it hangs as a loop. At unequal intervals along the loop hang equal pieces of lead wire, eight in all. Use short pieces of string for this purpose. Make a rough sketch of the completed apparatus, numbering the lead weights 1 to 8 in the direction from *A* to *B*.

Now measure and record :—

(*a*) The depth of the point of attachment of each lead weight below the level of the points of support of the cord ;

(*b*) The length of each interval of the cord between *A* and **1**, **1** and **2**, . . . **8** and *B*.

From these observations calculate the tangent of the angle of slope of each segment of the cord. Write these results in correct order.

9. On a cord about 2·5 metres long make two knots each 40 cm. distant from the mid-point. Tie the cord to two points at the same level and 1·6 metre apart in such a manner that each supporting point is distant 0·8 metre from the nearer knot. Attach at each knot short pieces of string carrying weights W_1 and W_2.

First, let W_1 be 2 kilos, and give W_2 a series of values. At each of these values measure the depth of the W_2 knot below the level of the point of support of the cord. Plot the observations as a curve with depths as abscissæ and weights as ordinates.

Second, hang up in turn the given objects in place of W_2, and observe for each the depth of the knot when a suitable known weight is suspended at W_1. By aid of the curve determine the weight of each object.

10. Suspend the known weight W and the unknown weight X from a cord $ACDB$ in the manner shown. Make such measurements as you consider sufficient and necessary

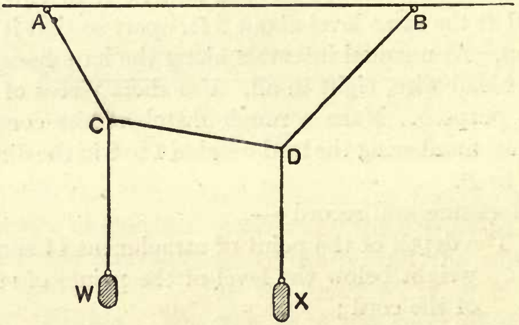


Fig. 234.

to fix the size and shape of the figure *ACDB*. Tabulate these measurements, and draw the figure *ACDB* to scale. Check your drawing by one or more additional measurements. State what checks you make. Determine graphically the value of *X*.

As a check on your result, determine independently the value of *X* by using the given beam as a ‘moment-beam’. Write a short description of what you do.

11. Determine the position of the centre of gravity of the given metal plate and mark it accurately on a piece of paper gummed to the plate.

Put a horizontal knife-edge through one of the holes in the plate and cause the plate to swing in its own plane. Adjust the length of the thread of a plumb-bob to have the same time of oscillation as the plate. Measure the distance between the knife-edge and the centre of gravity of the plate and also the length of the simple pendulum formed by the plumb-bob.

Make similar observations with five other holes of the plate, tabulate the results, and plot a curve exhibiting the connexion between the two lengths measured.

12. Allow the given light spring, carrying the given scale-pan, to hang vertically from a fixed pin. Place in the scale-pan in succession 0 gm., 25 gm., 50 gm., 75 gm., 100 gm. Note the extension produced in the spring beyond its natural length, and verify that the increase in the length of the spring is directly proportional to the load. [Take care not to stretch the spring much more than it is stretched when 100 gm. is placed in the scale-pan, lest you overstrain it.]

Determine the weight of the scale-pan. Give its number and weight.

Let W denote the total load supported, including the scale-pan. Gently raise the pan one or two cm. and let go. With the aid of a stop-watch determine the periodic time T sec. of oscillation. Obtain a set of corresponding values of W and T and tabulate your results. Also draw a graph exhibiting T as a function of W.

What value of W will make the periodic time 1 sec.? What extension would be produced in the spring beyond its natural length if it carried such a load in equilibrium?

13. A weight of 1 kilo is attached to each end of a cord passing over two fixed pulleys. You are to add from 100 to 300 gm. to the load on one side so that the larger weight can fall in an observed time through a distance to be measured in cm. State the actual load used.

Write down the relation between the time, the distance fallen, and the acceleration, and so find the acceleration on the C.G.S. system. Hence find the velocity of the weights at the instant when the heavier has descended 100 cm., and the total kinetic energy of the two weights. Find how much energy has been absorbed by friction, &c. when the heavier weight has descended 100 cm. [$g=981$ cm. per sec. per sec.]

14. Set up the groove at a convenient slope. Observe the time the ball takes to run down the groove, and give its final velocity in cm. per sec.

From measurements of the ball and the groove express the angular velocity of the ball in terms of its linear velocity.

The kinetic energy of a ball of weight m gm. and radius a cm. moving at v cm. per sec. and spinning at ω radians per sec. is $\frac{m}{981}\left(\frac{1}{2}v^2+\frac{1}{5}a^2\omega^2\right)$ gramme-centimetres. Calculate the kinetic energy of the ball on leaving the groove.

Give also the loss of potential energy, and account for any difference between this and the kinetic energy.

15. Arrange the pulleys as shown in the sketch. What is the velocity-ratio ?

Including the weight of the hook H as part of the effort required to raise the load W, determine the mechanical efficiency η when W is 100 gm. Vary the load W, giving it in succession the values 200, 300, 400, 500 gm. and determine a set of corresponding values of W and η. Tabulate your results. Also draw a graph exhibiting η as a function of W.

Draw a second graph exhibiting $\frac{1}{\eta}$ as a function of $\frac{1}{W}$, and hence obtain a formula expressing η in terms of W.

If you have not sufficient time to try all five values of W omit one or even two, but do not omit the last paragraph of the question.

16. Call the three strings knotted together OA, OB, OC. Fix A to a screw-eye near the left-hand end of the top-rail, and a weight of 3 kilos to B, and tie the end C to the hook of a spring balance. Attach a cord to the ring of the balance, and fasten the cord by a clip on one of the vertical pillars so that OC is horizontal. Measure the tension T indicated by the spring balance, and the angle θ which AO makes with the vertical.

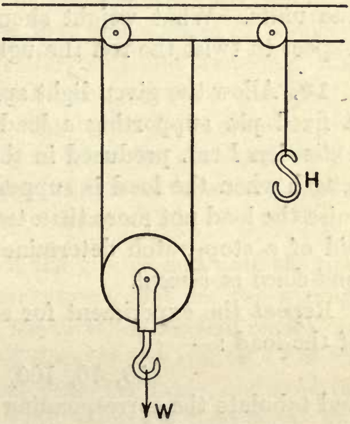


FIG. 235.

Vary the position of OC, and so the value of T, keeping OC always horizontal, and for each position observe the values of T and θ. The load of 3 kilos should not be changed. Give a graph co-ordinating T and $\tan \theta$. Under what circumstances would the tension in OA be a minimum, and what would be the minimum value ?

17. Hang various weights in succession from the pulley. In each case measure the angle through which the end of the steel rod is twisted. Tabulate your results, giving also the diameter of the pulley ; and give a graph of the angle of twist as a function of the weight.

Assuming the ratio angle turned through ÷ weight to be constant, give its value from your graph, stating the units. What weight should you (from your results) expect to twist the rod through 42° ? Verify by trial.

18. Allow the given light spring to hang vertically from a fixed pin supporting a load of 150 gm. Observe the extension l cm. produced in the spring beyond its natural length when the load is supported in equilibrium. Gently raise the load not more than two cm. and let go. With the aid of a stop-watch determine the time T sec. of *an up-and-down movement.*

Repeat the experiment for each of the following values of the load :—

50, 70, 100, 120 gm.,

and tabulate the corresponding values of T and l. In each case calculate the value of T^2/l, and give the average of your results.

Show by means of a graph how the value of T depends upon l. Read off the value of l which makes T equal to 1 sec.

19. Suspend the bullet by means of a thread and set it swinging as a pendulum. The length (l cm.) of the pendulum may be taken to be the distance of the centre of the ball from the point of suspension of the thread. Arrange so that the length of the pendulum is intermediate between 50 and 150 cm. Determine by observation the times (T sec.) of *a one-way swing* for 5 or 6 different values of l. Tabulate your results. In each case calculate the value of T^2/l and give the average of your results.

Show by means of a graph how the value of T depends upon l. Read off the value of l which makes T equal to 1 sec.

20. Measure the diameter of the given ball, explaining your method of measurement, and calculate its volume in cub. cm. You may assume that the volume of a sphere is $\frac{1}{6}\pi \times (\text{diameter})^3$. Weigh the ball, and assuming that 1 cub. cm. of water weighs 1 gm., calculate the specific gravity of the material of the ball. Determine experimentally what weight of iron must be attached to the ball so as just to sink it when it is placed in water. Hence calculate the specific gravity of iron.

21. Let the ball roll down the groove and note the time it takes. Take several observations, and use the average as the true time. Give also the distance the ball runs along the groove, and the angle of slope of the groove. Let the ball run down the groove and along the table, and note the time for the whole run. Take several observations and average them. Give also the distance the ball runs along the table. From the distance the ball runs along the groove, and the time it takes, calculate the acceleration of the ball, supposed uniform. Also find the speed along the table (supposed uniform); and, supposing this to be the speed on leaving the groove, use this and the distance run on the groove to calculate the acceleration along the groove. If time allows, repeat the whole experiment, letting the ball start some little distance from the top of the groove.

22. Arrange the inclined plane so that the cotangent of its inclination θ to the horizontal is 6. Explain how you do this.

Suspend a plumb-line (which should clear the table) from the point B (see diagram below), mark the point E of the floor vertically below B, and measure BE in cm.

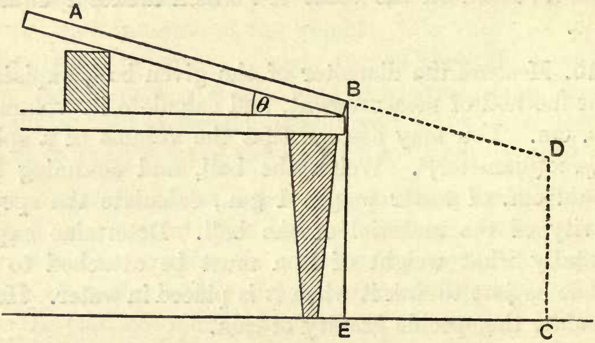


FIG. 236.

Take $AB = 150$ cm., and allow the steel ball to roll down the inclined plane from rest at A. If necessary pack the inclined plane, by placing under it one or more pieces of paper, so that the ball may not roll off the inclined plane till it comes to the bottom. Place a card, on which is spread a little plasticine, on the floor to receive the ball at C. Measure EC in cm.

Let the vertical through C meet AB produced in D. *Calculate* the lengths of BD and DC, and hence of $BD^2/4DC$. It can be shown that $BD^2/4DC$ is the height from which a body must be allowed to fall from rest to acquire a velocity equal to the velocity with which the centre of the ball was moving at the instant when it left the inclined plane at *B*. Calculate this velocity in cm./sec.

Assuming that the centre of the ball moved with uniform acceleration from *A* to *B*, calculate this acceleration in cm./sec^2. If you have time, take $AB = 120$ cm., and repeat the experiment, keeping the inclination θ the same as before.

23. Fig. 237 is the trace made by a pen attached to a spring which vibrated backwards and forwards in $\frac{1}{5}$ sec., while a strip of paper was drawn under it. The straight line in the figure was made by keeping the pen still while the paper was drawn back again.

From measurements of the figure show in a graph the whole distance moved by the strip of paper up to any time. Take 1 cm. to represent $\frac{1}{5}$ sec., and show the distance full size.

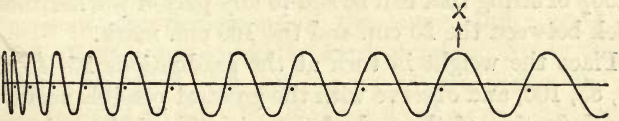


Fig. 237 (reduced).

From the figure measure the distance moved in $\frac{1}{5}$ sec. at various times, and use it to give a graph of the velocity of the strip of paper at any time. Represent $\frac{1}{5}$ sec. by 1 cm., and a speed of 1 cm. per sec. by 1 cm.

Find the average value of the acceleration, and taking this as the true value throughout, calculate the velocity of the strip of paper at the moment when the pen was at *X*. Check this value for the velocity by measuring the distance travelled in a short interval just before and in a short interval just after.

24. Tie one end of a string to the rim of the pulley, the other to a weight of 40 gm. Wrap the string on the pulley, and let the weight set the pulley in motion, and observe the time taken by the weight to drop 160 cm. Deduce the terminal speed of the weight, and find the corresponding angular velocity of the pulley in radians per sec. From measurements of the pulley rim (neglecting the rest of the pulley) find the moment of inertia of the pulley, and its terminal kinetic energy. The density of the rim is 7·2 gm. per c.cm.

How much energy has the weight lost, and how much has the pulley stored ? Account for any difference.

25. Clamp a metre stick horizontally to the table with 10 cm. of its length on the table and 90 cm. projecting free. Fix another metre stick vertically in such a position as to enable accurate readings to be made of the deflections of the free end of the horizontal stick. Tie a weight on a loop of string that can be slid to any part of the horizontal stick between the 25 cm. and the 100 cm. mark.

Place the weight in turn at the graduations 25, 40, 55, 70, 85, 100, and observe with the greatest possible accuracy the deflection of the end of the stick (the 100 mark) produced by the weight at each position.

Tabulate the observations and make a graph, using as abscissæ the distances of the weight from the edge of the table, and as ordinates the corresponding deflections of the end of the stick. The graph will be of the form $y = Ax^2 + Bx^3$. Determine A and B by substituting in this equation the co-ordinates of two points of the graph.

26. Measure the length of the brass wire between the supports. Call this $2a$. Suspend weights of 0·5, 1, 1·5, 2, 2·5 kilos in turn from the middle point of the brass wire, and measure the deflections (x) thereby produced. Compute the tension (T) in the wire corresponding to each load. Determine the area (α) of the cross-section of the wire, whose diameter you may assume to be 0·93 mm.

Assuming that the modulus of elasticity E is given by the formula

$$E = \frac{T}{\alpha} \times \frac{\text{original length}}{\text{extension in length}}$$

and that x/a is small, obtain an approximate expression for E in terms of the suspended weight W, and α and x/a. Draw a graph of the theoretical approximate relation between log W and log x, and mark on the graph the experimental values of log W and log x. State briefly what you consider to be the principal sources of error in the experiment.

27. Fix two lengths of wire horizontally between the table uprights, one about 50 cm. above the table and the other a little higher. The lower one is to be deflected by weights ; the upper wire merely affords a zero from which to measure the deflections of the lower, and should be loaded slightly by a middle fixed weight. Apply loads centrally to the lower wire so as to get deflections of between 1 and 10 cm. In all about 6 values of load should be used, and observations of the deflections should be made both while the load is being increased and while it is being decreased.

From the observations calculate the tension in the wire and the stretch of the wire at each observation. Tabulate these results, and from them obtain a mean value of Young's modulus of elasticity.

	0	1	2	3	4	5	6	7	8	9
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743
11	04139	04532	04922	05308	05690	06070	06446	06819	07188	07555
12	07918	08279	08636	08991	09342	09691	10037	10380	10721	11059
13	11394	11727	12057	12385	12710	13033	13354	13672	13988	14301
14	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140
16	20412	20683	20952	21219	21484	21748	22011	22272	22531	22789
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646
19	27875	28103	28330	28556	28780	29003	29226	29447	29667	29885
20	30103	30320	30535	30750	30963	31175	31387	31597	31806	32015
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090
29	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379
32	50515	50651	50786	50920	51055	51188	51322	51455	51587	51720
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995
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41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221
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43	63347	63448	63549	63649	63749	63849	63949	64048	64147	64246
44	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225
45	65321	65418	65514	65610	65706	65801	65896	65992	66087	66181
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117
47	67210	67302	67394	67486	67578	67669	67761	67852	67943	68034
48	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159
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26	53	79	105	132	158	184	210	237	16
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10	20	31	41	51	61	71	82	92	42
10	20	30	40	50	60	70	80	90	43
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8	16	24	32	40	48	56	64	72	54

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85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399
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91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802
93	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182
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97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078
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99	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957

1	2	3	4	5	6	7	8	9	
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6	12	18	24	30	35	41	47	53	73
6	12	17	23	29	35	41	46	52	74
6	12	17	23	29	35	41	46	52	75
6	11	17	23	29	34	40	46	51	76
6	11	17	22	28	34	39	45	50	77
6	11	17	22	28	33	39	44	50	78
6	11	17	22	28	33	39	44	50	79
5	11	16	22	27	32	38	43	49	80
5	11	16	21	27	32	37	42	48	81
5	11	16	21	27	32	37	42	48	82
5	10	16	21	26	31	36	42	47	83
5	10	15	20	26	31	36	41	46	84
5	10	15	20	26	31	36	41	46	85
5	10	15	20	25	30	35	40	45	86
5	10	15	20	25	30	35	40	45	87
5	10	15	20	25	29	34	39	44	88
5	10	15	20	25	29	34	39	44	89
5	10	14	19	24	29	34	38	43	90
5	9	14	19	24	28	33	38	42	91
5	9	14	19	24	28	33	38	42	92
5	9	14	18	23	28	32	37	41	93
5	9	14	18	23	28	32	37	41	94
5	9	14	18	23	27	32	36	41	95
5	9	14	18	23	27	32	36	41	96
5	9	14	18	23	27	32	36	41	97
4	9	13	18	22	26	31	35	40	98
4	9	13	17	22	26	30	34	39	99

Angle.	Circular measure.	Sine.	Tangent.	Cotangent.	Ccsine.		
0°	0	0	0	∞	1	1·571	90°
1°	·017	·017	·017	57·29	1	1·553	89°
2°	·035	·035	·035	28·64	·999	1·536	88°
3°	·052	·052	·052	19·08	·999	1·518	87°
4°	·070	·070	·070	14·30	·998	1·501	86°
5°	·087	·087	·087	11·43	·996	1·484	85°
6°	·105	·105	·105	9·514	·995	1·466	84°
7°	·122	·122	·123	8·144	·993	1·449	83°
8°	·140	·139	·141	7·115	·990	1·431	82°
9°	·157	·156	·158	6·314	·988	1·414	81°
10°	·175	·174	·176	5·671	·985	1·396	80°
11°	·192	·191	·194	5·145	·982	1·379	79°
12°	·209	·208	·213	4·705	·978	1·361	78°
13°	·227	·225	·231	4·331	·974	1·344	77°
14°	·244	·242	·249	4·011	·970	1·326	76°
15°	·262	·259	·268	3·732	·966	1·309	75°
16°	·279	·276	·287	3·487	·961	1·292	74°
17°	·297	·292	·306	3·271	·956	1·274	73°
18°	·314	·309	·325	3·078	·951	1·257	72°
19°	·332	·326	·344	2·904	·946	1·239	71°
20°	·349	·342	·364	2·747	·940	1·222	70°
21°	·367	·358	·384	2·605	·934	1·204	69°
22°	·384	·375	·404	2·475	·927	1·187	68°
23°	·401	·391	·424	2·356	·920	1·169	67°
24°	·419	·407	·445	2·246	·914	1·152	66°
25°	·436	·423	·466	2·145	·906	1·135	65°
26°	·454	·438	·488	2·050	·899	1·117	64°
27°	·471	·454	·510	1·963	·891	1·100	63°
28°	·489	·469	·532	1·881	·883	1·082	62°
29°	·506	·485	·554	1·804	·875	1·065	61°
30°	·524	·500	·577	1·732	·866	1·047	60°
31°	·541	·515	·601	1·664	·857	1·030	59°
32°	·559	·530	·625	1·600	·848	1·012	58°
33°	·576	·545	·649	1·540	·839	·995	57°
34°	·593	·559	·675	1·483	·829	·977	56°
35°	·611	·574	·700	1·428	·819	·960	55°
36°	·628	·588	·727	1·376	·809	·942	54°
37°	·646	·602	·754	1·327	·799	·925	53°
38°	·663	·616	·781	1·280	·788	·908	52°
39°	·681	·629	·810	1·235	·777	·890	51°
40°	·698	·643	·839	1·192	·766	·873	50°
41°	·716	·656	·869	1·150	·755	·855	49°
42°	·733	·669	·900	1·111	·743	·838	48°
43°	·750	·682	·933	1·072	·731	·820	47°
44°	·768	·695	·966	1·036	·719	·803	46°
45°	·785	·707	1·000	1·000	·707	·785	45°
		Cosine.	Cot.	Tangent.	Sine.	Circ. meas.	Angle.

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MECHANICS

ANSWERS

9. (i) You fall (*a*) backwards, (*b*) forwards.

(ii) Outside. (iii) The outside wheel.

(iv) Be ready to move your feet in the direction in which the vehicle is moving; otherwise you fall in that direction.

(v) The back.

(vi) A 'shove-off' is necessary in all cases.

(vii) The skater has no advantage in starting; but the impetus he acquires is not destroyed rapidly, as is the case with the runner.

(viii), (ix), (x) See § 124.

12. (i) 22·35. (ii) 45·76 sec.; 3 min. 3 sec.

(iii) 20·45; 548·6.

13. (i) 2·828 miles NE. (ii) 11·1 miles, E. 63·5° N.

(iii) $\dfrac{88x}{y}$. (iv) 14′ 8·5″ per hour.

14. (i) 13 miles an hour, at 23° with the rails.

(ii) $10\frac{5}{8}$ miles a day, W. nearly.

16. (iii) 8·95 . . miles an hour, E. 63·4° N.

(iv) Let *AB* represent velocity of train. Draw *AC* perpendicular to *AB*, and with centre *B* and radius representing the speed of the bullet describe an arc cutting *AC* at *C*. *BC* is the required direction.

(v) NE., 0·637 miles per hour.

(vi) Path slants backwards, angle 14°.

(vii) $\sin^{-1}\dfrac{u}{v}$ behind object.

(viii) *A* goes round in 2π seconds, *B* in $\dfrac{2\pi}{3}$ seconds. Therefore *B* gains 8 right angles in 2π seconds, and 2 right angles in $\dfrac{\pi}{2}$ seconds.

(ix) 38°.

(x) Draw concentric circles every half-inch to 8 in., and eight radial lines every 45°. The fly moves $\frac{1}{2}$ in. outwards and the disc rotates 45° in each quarter-minute. Final velocity 25·4 in. per min.

(xi) 10 miles per hr. from W. approximately.

(xii) 20 miles per hr., E. 37° S.; 6 miles.

18. (i) $\frac{QN}{NT} = \frac{\frac{1}{8}\text{ mile}}{\frac{1}{4}\text{ minute}} = 30$ miles an hour.

(ii) The points are not quite in a straight line; therefore the velocity is not quite uniform.

(iii) 8, 16, 24, 32 ft. per sec.

19. (i) 40, 112 (ft. per sec.).

(iii) 8 ft. per sec. per sec.; 32 ft. per sec.; 40 ft. per sec.; 36 feet.

(iv) (*a*) $\frac{1}{2}$; (*b*) $\frac{11}{15}$. (v) 196 feet.

(vi) $3\frac{1}{2}$ seconds; 112 ft. per sec.

20. (iii) See §§ 28, 29.

(iv) 2 ft. sec. sec.; 25 ft.; 75 ft.

(vii) $2\frac{1}{2}$ sec.; 100 ft.; 80 ft. per sec.

22. (i) (*a*) 228; (*b*) 164 ft. per sec.; (*c*) 656 ft.

(iv) Let u = initial velocity; velocity at highest point = 0.

t = time of ascent $= \frac{u}{g}$ (since $u - gt = 0$).

$$\therefore \text{ Greatest vertical height} = ut - \tfrac{1}{2}gt^2 = \frac{u^2}{g} - \frac{1}{2}\frac{u^2}{g} = \frac{u^2}{2g}.$$

$$\text{If } t' = \text{time of descent, then } \frac{u^2}{2g} = \tfrac{1}{2}gt'^2.$$

$$\therefore t' = \frac{u}{g} = t.$$

Velocity on return $= -u$.

(v) See § 39.

(vi) $v = \frac{ds}{dt} = u + at$; $\frac{dv}{dt} = a$.

(vii) 16, 48, 80, 112 feet. (viii) 625 feet; $6\frac{1}{4}$ seconds.

(ix) 64, 96, 96, 64, 0 ft. above starting-point; greatest height 100 ft.; 40 ft. apart.

(x) 89 seconds. (xi) $25\frac{25}{44}$; $34\frac{1}{11}$; 2·5.

(xii) 1·112 sec. (xiii) $\frac{3}{8}$.

(xiv) 4·5 sec.; 320 ft. (xv) 3·536 sec.; 113 ft. per sec.

(xvi) 173 metres. (xvii) $-\frac{2}{3}$; $2\frac{2}{3}$ (cm.-second units).

(xviii) $293\frac{1}{3}$. (xix) The number would be divided by 4.
(xx) $-\frac{44}{45}$ ft. sec. sec. (xxi) 2·2 ft. sec. sec. ; 440 feet.
(xxii) 32 ft. per sec. (xxiii) $\frac{3}{4}$ mile.
(xxiv) 0 ; 3 min. 20 sec. (xxv) $\frac{2u}{f}$.

(xxvi) 3, 2, 6 ft. from O ; 21 ft. ; 7 ft. per sec. ; ·004001 ft. ; 4·001 ft. per sec. ; 4 ft. per sec.

(xxvii) 14 metres per sec.

(xxviii) See § 40 ; 40 ft. per sec.

50. (i) $\frac{1}{15}g$; 21,974 ; 22·4.

(ii) $\frac{m}{M+m}g$; $\frac{Mm}{M+m}g$.

(iv) 6,000 dynes. (v) 4 cwt.

(vi) 23 lb. weight nearly.

(vii) 304,800 lb. foot-sec. units ; 635 lb. weight.

(viii) 3 ; $159\frac{3}{8}$ lb. ; $140\frac{5}{8}$ lb.

(ix) 4,896 (taking 1 pound moving 1 ft. per sec. as unit of momentum).

(x) (*a*) Yes ; (*b*) No, if the acceleration is upwards.

(xi) 1 ft. sec. sec. (xiii) $\frac{9}{8}W$ upwards ; W.

(xiv) $\frac{Wv}{W'g}$; 11 seconds. (xv) $4\frac{7}{12}$ tons.

(xvi) $\frac{Q-R}{P+Q+R}\cdot g$; 7 lb. ; 32 ft. sec. sec.

(xvii) (*a*) 15 grains weight. (*b*) At 60° with downward vertical.

(xviii) 2 ft. sec. sec. ; $15\frac{15}{16}$ oz. ; $14\frac{1}{16}$ oz. ; $13\frac{13}{16}$ oz.

(xx) 10 metres ; 960,000 dynes. (xxi) 3.

51. (i) 13,835.

53. (i) 1,793 ft. per sec.

57. (i) The pull of the traces pulls the cart forward and the horse back. The pull of the ground on the horse's feet is greater than this, and therefore both horse and cart move forward. Disregarding friction of wheels, &c., the force required for horse and cart is double of that required for either separately.

(ii) 83·2 cm. per sec.

(iii) 0·89 ft. per sec. ; nearly 7 cwt.

(iv) The balance reading would be increased.

(v) 2·6 lb. weight.

(vi) The balance moves up to meet the weight, and acquires a momentum, the moment of which about the knife-edge is nearly equal to that of the weight itself. When they meet, the momenta destroy each other. (See p. 231, note.)

(vii) 6·1 ft.

(viii) The monkey pulls up the weight as well as itself; friction will prevent the weight from ascending as fast as the monkey.

63. (i) $\sqrt{P^2+Q^2}$. (ii) $P+Q$. (iii) $P-Q$.
(iv) 12·2 lb. N. 55° E. (v) (1) 18·8. (2) 45·5. (3) 11·7.

67. (i) $\frac{12}{13}$ cwt.; $\frac{5}{13}$ cwt. (ii) $P = W \tan \alpha$; $T = \dfrac{W}{\cos \alpha}$.
(iii) 79·18 lb., 45° from vertical.
(iv) 25 lb.; 111° from direction of smaller force.
(vi) 91 lb.; 77 lb.
(viii) $\dfrac{W}{2 \sin \alpha}$; No (see § 75). (ix) 4 lb.; 54°.
(x) 43 lb.; 51°. (xi) 17·3 lb.; 90°.
(xii) 1·73; 80° from OA. (xiii) 16 lb.; 20 lb.
(xiv) 60°. (xv) 289 grammes weight.

69. (i) 3·64 lb.; 10·64 lb.

71. (i) 2·2 tons; 2 tons.

(ii) 5 tons 5 cwt.; 2 tons 7 cwt. If the weight were fastened to B, these forces would become $4\frac{1}{6}$ tons, $3\frac{1}{3}$ tons.

(iii) 16 lb.; 12 lb. (iv) 70 lb.

72. (i) 320 lb., 680 lb. (ii) 12 lb. (iii) 300 lb.
(iv) 134 lb.; 139°. (v) 4 lb., making 50° with the 5 lb.
(vi) $W \sin \alpha$, $W \cos \alpha$; 45°. (vii) $6\frac{2}{3}$, 9·43, 5·77 lb.
(viii) 30 lb. tension; 77 lb. pressure.

73. (i) 16 cm. (ii) 25 grammes.
(iv) 40 cm.; $50 \times 20 = 20 \times 30 + 10 \times 40$.

77. (iii) 40 lb.; 40·5 lb.
(iv) 1 ft. 6 in. from the 7 lb. weight. (v) 7·5 lb., 12·5 lb.
(vii) $\frac{1}{2}$ cwt. at right angles to CA.

83. (i) 7·68 in. from scale-pan.

84. (ii) 80 lb. (iii) 3 lb. downwards, 6 ft. from the 1 lb.
(iv) 1·6 lb. (v) 120 lb.; 40 lb.
(vi) 16 turns nearly; 12 lb. nearly. (vii) 430·495 grammes.
(viii) 140 lb.; the resultant passes through the fulcrum.
(ix) 2 lb. (xi) $\frac{9}{11}$ in. from the 10 lb. weight.
(xii) 0·15 in.; 30 lb. (xiii) $5\frac{1}{4}$ tons, $1\frac{3}{4}$ tons.
(xiv) 7 lb., $14\frac{2}{7}$ ft. outside larger force. (xv) 1 foot.

(xvi) 3 lb. (xvii) $2w$, at middle point of AC.

(xviii) 91 lb. (xix) (a) $W \cdot \frac{BP}{AB}, W \cdot \frac{AP}{AB}$. (See p. 250.)

88. (iii) If AD is the median on which the centres of gravity lie, the C. G. of the remainder is distant $\frac{2}{9}$ of AD from D.

(iv) $\frac{b^2}{a+b}$ from centre of larger circle.

89. (i) It must just exceed W, and be placed on the edge midway between two supports.

91. (i) Let D, E, F be the middle points of the sides a, b, c; then distance of centre of gravity from DE

$$= \frac{c}{a+b+c} \times \tfrac{1}{2} b \sin A = \frac{\Delta}{2s} = \frac{r}{2}.$$

∴ C. G. lies at in-centre of Δ DEF.

1·5 in., 5 in.

(ii) Let the side of the square $= 1$; the centre of the parallel forces lies on a line half-way between the 6, 10 line and the 14, 18 line, at a distance $2\frac{1}{2}$ outside the 10, 18 line.

(iv) One inch from the common base, on the side of the larger triangle.

(v) On diagonal, $\frac{1}{12}$ of its length from the centre of the original square.

(vi) $2\frac{13}{23}$ in. from the wider end.

(vii) 2·4 in. from centre of larger circle.

(viii) Bisect BC at F. Join AF, EF, and find G_1, G_2, the centres of gravity of ABC, EBC. Join $G_1 G_2$ cutting BC at H, and cut off $G_2 G$ equal to $G_1 H$. G is the point required.

(ix) See Fig. 124.

(x) If the sides of the rectangle are a, b, and the side of the square c, the distance of the C. G. of the remainder from the centre of the rectangle is $\frac{c^2(b-c)}{2(ab-c^2)}$.

(xi) Distance from centre of square $= \frac{2}{15}$ of one side.

(xii) 65 lb. 10 oz.; $9\frac{2}{7}$ in., $13\frac{1}{3}$ in.

(xiii) 1 : 18. (xiv) $7\frac{1}{2}$ in. from the 11 lb. weight.

(xv) Over 525 lb. weight.

(xvii) The 1 lb. weight must be (1) vertically below, (2) vertically above the point of support. If the point of support were half-way between the 1 lb. weight and the intersection of medians, the equilibrium would be neutral.

92. (i) 9 lb.; 6 lb. at 27° with horizontal. (ii) 11·53 lb.
(iii) $5\sqrt{3}$ lb.; $5\sqrt{7}$ lb., 41° from vertical.

97. (i) $4\sqrt{3}$ lb.

(ii) Let 1 cm. represent 1 oz.; draw *LM*, *MN*, *NO* vertical to represent the weights 2 oz., 10 oz., 2 oz.; from *L*, *O* draw *LP*, *OP* perpendicular to the directions *AB*, *AC*, to represent the reactions of the rods on the rings. Then *PM*, *PN* give the directions of the threads joining the weights.

(iii) $5\frac{1}{2}$ tons.

(iv) Draw *PQ* vertical to represent 1,000 lb.; from *Q* draw *QO* parallel to *AB*, from *P* draw *PO* parallel to *BC*. Then *QO*, *OP* represent the thrusts in *AB*, *BC*. Draw *OR* horizontal to meet *PQ* in *R*. Then *OR* represents the pull in *AC*. Likewise *QR*, *RP* represent the reactions at the supports.

(v) 55, 60, 30, 75 grammes.

98. (i) $13{\cdot}554 \times 10^6$. (ii) 98,100,000; 7·2.

108. (ii) 10·85 ft. (iii) The same.

(v) 4·4 ft. per sec.

(vi) Resolved force × distance = change in kinetic energy; $937\frac{1}{2}$ lb.

(vii) 15,769,600; the mass moved, and the acceleration.

(viii) 25,000. (ix) (*a*) Correct; (*b*) too small.

(x) $\dfrac{\text{Energy of iron}}{\text{Energy of bullet}} = 1{\cdot}43$; $\dfrac{\text{momentum of iron}}{\text{momentum of bullet}} = 143$.

(xi) $5{\cdot}12 \times 10^{10}$ ergs. (xii) K. E. 36, 0, 100 foot-pounds.
P. E. 64, 100, 0 foot-pounds.

(xiv) 5,820 foot-pounds.

(xv) 7·2 ft. per sec.; the velocity will be less in the second case, as the 7 lb. weight will have been raised a little.

(xvii) 500 foot, lb., second units; 5·58 foot-pounds.

(xviii) 38·3 ft. sec. sec.; 100 pounds weight nearly.

(xix) 21 pounds weight.

(xx) (*a*) No, because there is no displacement.

(*b*) No, because the displacement is always at right angles to the direction of the force; the slight movement of the centre necessary to start and maintain the motion being disregarded.

(*c*) Yes, if the engine is exerting any force on the train.

(xxii) 30,250 foot-tons; 5,252 foot-tons.

(xxiii) 5·07 tons weight; 133,861 foot-tons.

(xxiv) Whole work done = *Rh*.

115. (i) 20°; 4.6 lb. (ii) 2,150. (iii) 12 miles per hr.
(iv) 31°; 8.32 ft. per sec.; 33 ft. per sec.
(v) 344. (vi) 13°. (vii) 12°.
(viii) Consider the velocity of the steamer relative to the body from which it obtains a reaction.

117. (ii) 4 : 1; 6 : 1. (iii) 2 : 1; 1 : sin θ.

121. (ii) When the radius bisecting the angle between the two weights is horizontal; $\frac{Rw}{r}$.
(iv) $\frac{1}{4}W$; $\frac{1}{5}W$. (v) $W = 4P$; $60\frac{1}{4}$ lb.
(vi) 18·5, 51, 64 per cent. (vii) 100 lb., 250 lb.
(viii) The tensions are 132, 68, 36, 20 lb.
(ix) If R, r_1, r_2, are the radii of the wheel and the two parts of the axle, the velocity ratio is $\frac{2R}{r_1 - r_2}$. See p. 205.
(x) 9·7 lb.; 26·3 lb. (xi) 10 lb. nearly; 28·3 lb.
(xii) 7·8, 35·1, 54·1, 56·3 per cent.

122. (i) 692·4 ft.; 206·3 ft.
(ii) 104 yds., 90 yds., 90 yds. Max. height 78 ft., 117 ft., 39 ft.
(iii) 217 metres.
(iv) The velocity at any instant is the resultant of two vectors, $V \cos \alpha$ horizontally, and $V \sin \alpha - gt$ vertically. Hence the ends of the lines all lie on a vertical straight line. After $3\frac{1}{2}$ seconds the velocity is 4° below the horizontal.

123. (iii) $\frac{2V^2 \sin(\alpha + \beta) \cos \alpha}{g \cos^2 \beta}$.
(v) After 4 seconds, 512 ft. from base of tower, at 45° with horizontal.
(vi) See § 123 (*c*).
(vii) β is given by the equation

$$gh(\tan^2 \beta + 1) - 2V^2 \tan \alpha (\tan \beta - \tan \alpha) = 0.$$

(ix) From the equation $v^2 = u^2 - 2gh$ we get

$$v = 1497{\cdot}43 \text{ ft. per sec.}$$

124. (i) 600,000 dynes. (ii) 171·5 cm. per sec.
(iv) 14 miles an hour. (v) 25·3 ft. per sec.

125. (i) (*a*) 1·407 sec.; (*b*) 1·414 sec. (iv) 39·1 inches.

126. (i) 30 feet. (ii) 19°. (iii) 2·6°.

127. (i) (*a*) $\frac{\pi}{1800}$; (*b*) $\frac{\pi}{21600}$ (radians per second).

(ii) 4,200 ; 0·26.

(iii) ·00958 radians per hour; ·00897 (foot-second units), i. e. about $\frac{1}{3600} g$.

129. (ii) 3,072 ; ·00744. (iii) $4 \pi^2 n^2 r$.

(iv) ·00892 ft. sec. sec. ; $4{\cdot}74 \times 10^{20}$ lb. weight.

(v) 12 ft. per sec.

135. (i) $\frac{31}{256}$. (ii) 1 : 0·95.

144. (i) Bending moment = 84 (foot-pound units); shearing force = 0.

153. 8,012 metres. (1 litre of air weighs 1·29 grammes.)

157. (iii) 34 feet. (iv) 10 cc. (v) 5·8 lb.

(vii) The pressure on the under side of the piston is the atmospheric pressure *less* 24 feet of water. Hence the difference in the pressures, above and below the piston, against which work has to be done, is equal to 24 feet of water. This makes the force to be applied to the handle 8·32 lb. Work done = 2,400 foot-pounds.

(viii) Up stroke $34\frac{2}{3}$ lb. ; down stroke $138\frac{2}{3}$ lb.

166. (i) A force equal and opposite to the weight of water displaced.

(ii) The cork will displace less water. Weight of cork = weight of water displaced + weight of air displaced by the part of the cork above the water.

(iii) The aluminium must displace more air, and therefore loses more when weighed in air.

(iv) 350 grammes. (v) 1·47 in. See § 151.

(vi) 23·3 cm. ; 1. (vii) $\frac{8}{9}$.

(viii) A weight equal to that of 64 cubic in. of water must be added to the other pan. With spirit only 0·9 of this weight would be needed.

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